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Kvantové struktury

Quantum Structures

Summary

This paper presents and comments selected author's results in the theory of quantum structures.

Section 1 gives basic motivation and notions of quantum structures theory.

Section 2 deals with set representations of orthoposets and extensions of measures generalizing classical results for Boolean algebras (Stone [51], Horn and Tarski [20]).

Section 3 is devoted to Boolean orthoposets. Boolean orthoposets have a couple of properties close to properties of Boolean algebras. E.g., concreteness, distributivity of the MacNeille completion, weak form of distributivity, orthomodularity in case of finite orthocompleteness, unital set of two-valued states, equivalence of the unitality and order determinacy for a set of two-valued states.

Section 4 deals with conditions that ensures that a quantum structure is a Boolean algebra. Author generalizes quite a few results combining properties of the algebraic structure and of its state space for orthomodular posets and for effect algebras.

Section 5 treats Kochen–Specker-type constructions of orthomodular posets with 'small' state spaces.

Section 6 presents various results about classes of effect algebras that generalize both lattice and orthocomplete effect algebras.

Section 7 presents results concerning a characterization of atomistic effect algebras and a relation of orthoatomisticity to weak orthocompleteness.

Section 8 deals (including infinite sets) with the associativity of the partial operation $\oplus$ and with the distributivity of suprema and infima with respect to partial operations $\oplus$ and $\ominus$ and vice versa.

Souhrn

V této práci jsou uvedeny a komentovány vybrané autorovy výsledky z teorie kvantových struktur.

Část 1 uvádí základní motivaci a pojmy z oblasti kvantových struktur.

Část 2 se zabývá množinovými reprezentacemi ortoposetů a rozšiřováním měr, přičemž zobecňuje klasické výsledky pro Booleovy algebry (Stone [51], Horn a Tarski [20]).

Část 3 je věnována Booleovým ortoposetům, které mají řadu vlastností blízkých Booleovým algebrám. Například konkrétnost, distributivitu MacNeillova zúplnění, slabou formu distributivity, ortomodularitu v případě konečné ortoúplnosti, unitální množinu dvouhodnotových stavů, ekvivalenci unitality a určení uspořádání pro množinu dvouhodnotových stavů.

Část 4 se zabývá podmínkami, které zajistí, aby kvantová struktura byla Booleovou algebrou. Autor zobecňuje řadu výsledků vycházejících z vlastností algebraické struktury a z vlastností jejího stavového prostoru pro ortomodulární posety a pro efektové algebry.

Část 5 pojednává o Kochenových–Speckerových konstrukcích ortomodulárních posetů s malými stavovými prostory.

Část 6 obsahuje různé výsledky o třídách efektových algeber, které zobecňují jak svazové tak ortoúplné efektové algebry.

V části 7 jsou zmíněny výsledky týkající se charakterizace atomistických efektových algeber a vztahu mezi ortoatomisticitou a slabou ortoúplností v atomických efektových algebrách.

Část 8 se zabývá (včetně nekonečných množin) asociativitou částečné operace $\oplus$ a vzájemnou distributivitou suprema a infima na jedné straně a částečných operací $\oplus$ a $\ominus$ na straně druhé.

Klíčová slova

Booleova algebra, ortomodulární svaz, ortomodulární poset, ortoposet, Booleův ortoposet, efektová algebra, konkrétní, atomický, atomistický, ortoatomistický, kompatibilita, centrální prvek, (slabá) ortoúplnost, stav, Jauchův–Pironův stav, unitální množina stavů, množina stavů určující uspořádání, množinová reprezentace, vlastnost maximality.

Keywords

Boolean algebra, orthomodular lattice, orthomodular poset, orthoposet, Boolean orthoposet, effect algebra, concrete, atomic, atomistic, orthoatomistic, compatibility, central element, (weak) orthocompleteness, state, Jauch–Piron state, unital set of states, order determining set of states, set representation, maximality property.

Contents

1 Introduction

The origin of quantum structures was derived from the quantum physics formalism where the orthomodular lattice of closed subspaces of a Hilbert space was used. This enables a nonclassical (non-Boolean) logic with noncompatible elements. The abstract mathematical model interprets events as elements of some set. The physical state describes probabilities of measuring the events, i.e., the probability measure on the event structure, which is identified with the state.

Orthomodular lattices [24] and more general *orthomodular posets* [16, 45] are bounded posets (with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$) described by the properties of the partial ordering and the orthocomplementation. The crucial role plays the existence of orthogonal suprema (orthoposets) and the orthomodular law that is equivalent to the property that considering the interval up to a nonzero element we obtain structure of the same type.

Generalizations of this attempt uses (basically) the axiomatization of the partial operation $\oplus$ of orthogonal suprema (commutative, associative, with unique orthosupplement) such that the orthomodular law (in new terms) is fulfilled automatically. This leads to the notion of an *orthoalgebra* ($a \oplus a$ is defined only for $a = \mathbf{0}$) [12] and, more generally, of an *effect algebra* ($a \oplus \mathbf{1}$ is defined only for $a = \mathbf{0}$) [11, 10]. The latter has a physical interpretation as the set of effects, i.e., positive selfadjoint operators bounded by the identity on a Hilbert space.

An equivalent structure to the effect algebra is the so-called D-poset. It is defined using the axiomatization of the partial operation of the relative complement $\ominus$ [29, 10]. The relation between $\oplus$ and $\ominus$ is given by the equivalence of the following equalities: $a \oplus b = c$, $a = c \ominus b$, $b = c \ominus a$.

States are probability measures. *Jauch–Piron* states [22, 41] (motivated by physics) are such states that the set of elements evaluated by 1 is downward directed. We usually want to have a 'large' set of states, e.g., *order determining*, i.e., the partial order on the quantum structure is defined by values of states, or *unital*, i.e., every nonzero element has probability 1 for some state.

2 Set representations of orthoposets

Since the Stone representation [51] of Boolean algebras (by means of clopen subsets of totally disconnected compact Hausdorff topological spaces) it has been natural to look for a set (or even a topological) representation of algebraic structures in the sense that the basic objects of the algebraic structure (elements, relations, operations) are represented by set-theoretical ones. E.g., every Boolean algebra might be represented by a family of subsets of some underlying set such that the least element corresponds to the empty set, the greatest element corresponds to the underlying set, the partial ordering corresponds to the inclusion relation, the orthocomplementation corresponds to the set-theoretical complement, the infimum (meet) corresponds to the intersection and the supremum (join) corresponds to the union.

The representation might be constructed by several equivalent ways. We can use different objects as the elements of the underlying set: ultrafilters, prime ideals, homomorphisms into the two-element Boolean algebra or two-valued measures. The natural object in some applications of the theory of orthostructures (e.g. in the quantum theory) is the two-valued state (two-valued probability measure).

There need not be an infimum or a supremum of a pair of elements in orthoposets. Hence, we would like to find a set representation of an orthoposet such that the least element corresponds to the empty set, the partial ordering corresponds to the inclusion relation, the orthocomplementation corresponds to the set theoretical complementation and the supremum of orthogonal elements corresponds to the set theoretical union. However, it is known that the existence of such a representation is equivalent to the existence of an order determining set of two-valued states (Zierler and Schlessinger [71] for orthomodular posets). Orthoposets with such property are called *concrete*. Since there are orthomodular lattices even without any state (Greechie [14]), not every orthomodular lattice is concrete.

Thus, it is necessary to give up the latter correspondence and look for a weaker one. The weakening uses some type of weak (two-valued) states, i.e., assuming additivity only for some pairs of orthogonal elements. The supremum of these elements corresponds to the union of the corresponding representations if there is an order determining set of such weak states. Instead of a topological space we obtain a more general closure space [5, 21].

We might consider no additivity at all. It leads to the concept of an M-base and a set representation of every orthoposet (Marlow [30], Katrnoška [25], Mayet [31]). 'Better' representations were obtained assuming the additivity in case: both elements belong to the center of the orthomodular poset (Zierler and Schlessinger [71], Iturrioz [21]), at least one element belongs to the center of the orthomodular poset (Pták [42], Binder and Pták [2]), both elements belong to a given Boolean subalgebra of the orthomodular poset (Tkadlec [53]), at least one element belongs to the center or both elements belong to a given Boolean subalgebra of the orthomodular poset (Harding and Pták [19]).

Also, the topological properties of the representation and the extension of states are studied. A general approach for various types of weak states in orthoposets was given by Tkadlec [55].

3 Boolean orthoposets

An orthoposet is called *Boolean* if it fulfills the following condition: $a \wedge b = \mathbf{0}$ implies $a \perp b$ (the reverse implication is always valid). Boolean orthoposets are close to Boolean algebras, e.g., they are concrete [36]. Moreover, lattice [36] or orthocomplete [56] Boolean orthoposets are Boolean algebras (a generalization of these two statements is presented in Section 4).

The author proved that the MacNeille completion of a Boolean orthoposet is a Boolean algebra [55] and that Boolean orthoposets are almost distributive in the sense that the equality $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ is true whenever the expressions in parentheses and at least one side of this equality exist [52]. The latter result generalizes the result of Klukowski [27] (he, moreover, assumed the orthomodularity and the existence of the right side of the equation). Analogous results might be found in [16, Lemma 3.7] (for an orthomodular poset with additional assumptions about the compatibility and that the right side of the equation exists) and in [45, Proposition 1.3.10] (for orthomodular lattices with additional assumptions about compatibility and that the left side of the equation exists). Consequently, a Boolean orthoposet with existing (finite) orthogonal suprema is orthomodular.

The author gave a characterization of Boolean orthoposets by means of two-valued states [54] as orthoposets with a unital set of two-valued states such that every unital set of two-valued states is order determining. Also, he proved that an orthoposet is Boolean if it has an order determining set of two-valued Jauch–Piron states while the reverse implication is not true in general (giving an example of a Boolean orthomodular poset without any two-valued Jauch–Piron state) but it is true for atomic orthoposets. Moreover, there are concrete orthomodular posets with a unital set of two-valued states that are not Boolean.

4 Central elements

Central elements are elements that are compatible with all elements, i.e., they 'cuts' other elements additivelly. The set of central elements (the center) of an effect algebra forms a Boolean subalgebra, an effect algebra is a Boolean algebra if and only if every its element is central, an effect algebra can be decomposed into a product of effect algebras using a central element (and its orthosupplement) [15].

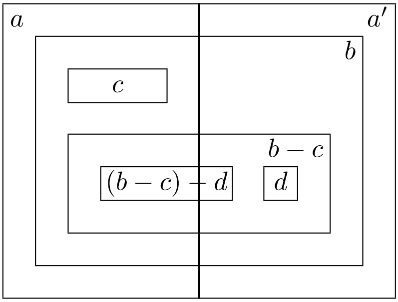
An important question in the axiomatics of quantum structures is the question whether the conditions imposed on the structure implies that the structure is a Boolean algebra (a classical system). In other words, we cannot combine such properties to obtain a proper quantum (nonclassical) structure.

Quite a few results of this type appeared for orthomodular posets combining a property of the orthomodular poset (*lattice* or *orthocomplete* or *Jauch–Piron with a countable unital set of two-valued states*) and a property of its state space (e.g., *order determining* / *unital* / *weakly unital* set of *two-valued Jauch–Piron* / *subadditive* / *weakly subadditive* states). In particular, the state space of a Boolean orthoposet has a suitable property.

Some of these results, e.g., the results of Dvurečenskij, Klukowski, Länger, Majerník, Müller, Navara, Pták, Pulmanová or Tkadlec [26, 36, 52, 34, 34, 48, 33, 47, 46, 56, 9, 57, 35] published during the years 1975–1997 were generalized by Tkadlec [58] using the introduced *maximality property* (every pair of elements has a maximal lower bound) and the weak distributivity ($a \wedge b = a \wedge b' = \mathbf{0}$ implies $a = \mathbf{0}$).

It is more complicated to prove that the maximality property and the weak distributivity is a consequence of properties used in the papers mentioned above than the proof of the theorem "Every weakly distributive orthomodular poset with the maximality property is a Boolean algebra." itself. The proof of the theorem might be illustrated using the set representation mentioned in

Section 2):



a, b
$c \leq a, b$ maximal
$b - c$ (orthomodularity)
$d \leq a', b - c$ maximal
$(b - c) - d$ (orthomodularity)
$\big((b-c)-d\big) \wedge a = \big((b-c)-d\big) \wedge a' = \mathbf{0}$ (maximality of c, d)
$(b - c) - d = \mathbf{0}$ (weak distrib.)
$(b - c) = d$
$b = c \vee d$, $a = c \vee (a - c) \ldots\ a \leftrightarrow b$

The result was generalized for effect algebras by Tkadlec [62] giving a characterization of central elements. Consequences for atomic effect algebras were presented by Tkadlec [63] and for residuated orthomodular lattices by Tkadlec and Turunen [69].

Other results stating when a quantum structure is a Boolean algebra are presented in author's papers [59] (for concrete orthomodular posets generalizing some results of Müller, Pták or Tkadlec [34, 44]) and [64] (for atomic sequential effect algebras generalizing some results of Gudder and Greechie [18]). The latter was generalized by Caragheorgheopol and Tkadlec [4] to effect algebras with compression bases [17].

5 Kochen–Specker-type constructions

Kochen and Specker [28] constructed an example of an orthomodular poset without any two-valued state using subspaces of a three-dimensional Hilbert space. They depicted their example by the so-called orthogonality diagram. Svozil and Tkadlec [20] used simpler Greechie diagrams [14] to present examples of orthomodular posets realizable in a three-dimensional Hilbert space with a 'small' (empty, not unital, not separating, not order determining) set of two-valued states. Also, the realizability of Greechie orthomodular posets was discussed. Several examples of Greechie diagrams and dual Greechie diagrams of such constructions of other authors (Schütte, Peres, Conway and Kochen, Mermin or Bub [7, 39, 40, 32, 3]) were presented in this and subsequent author's papers [60, 61].

6 Classes of effect algebras

There are various important classes of effect algebras.

Ovchinnikov [38] introduced *weakly orthocomplete orthomodular posets* (every orthogonal sum either has the sum or no minimal majorant) as a common generalization of orthocomplete orthomodular posets and orthomodular lattices and showed that they are disjunctive. Weak orthocompleteness is useful in the study of orthoatomisticity and disjunctivity might be used to characterize atomisticity (see Section 7). De Simone and Navara [8] introduced a stronger property denoted by $W+$ (the set of majorants of every orthogonal set is downward directed).

Tkadlec [58] introduced the class of orthomodular posets with the *maximality property* (every pair of elements has a maximal lower bound) as another common generalization of orthocomplete orthomodular posets and orthomodular lattices (lately for effect algebras) and showed various consequences of this property (see Section 4).

Tkadlec [66] introduced a stronger property denoted by CU (the set of upper bounds of every chain is downward directed) and showed that both lattice and orthocomplete effect algebras have this property. Moreover, he proved at this paper that a unital set of Jauch–Piron states on an effect algebra with the maximality property is strongly order determining and that a Jauch–Piron effect algebra with a countable unital set of states is an orthomodular lattice.

Author's papers [67, 68] show that weak orthocompleteness and the maximality property are incomparable in general as well as their strengthenings (properties W+ and CU) but there are some relations (in particular, properties W+ and CU are equivalent in separable Archimedean effect algebras) and they share common properties: the property W+ is a common generalization of the orthocompleteness and the lattice property; an orthomodular poset with the maximality property is disjunctive.

7 Atomic effect algebras

Atoms are minimal nonzero elements. An effect algebra is *atomic* if every its nonzero element dominates an atom, *atomistic* if every its element is a supremum of a set of atoms, and *orthoatomistic* if every its element is a sum of atoms.

Obviously, every orthoatomistic and every atomistic effect algebra is atomic, every orthoatomistic orthomodular poset is atomistic. Moreover, every atomic orthomodular lattice is atomistic (see, e.g., [45]).

Atomic orthomodular posets need not be atomistic (Greechie [13]), atomistic orthomodular posets need not be orthoatomistic (Ovchinnikov [38]) and orthoatomistic effect algebras need not be atomistic (Foulis, Greechie and Rüttimann [12]).

Author's papers [63, 64, 4] studying centrality in atomic effect algebras has been already mentioned in Section 4. It is also proved that every lattice effect algebra determined by atoms is atomistic [64].

A characterization of atomistic effect algebras as disjunctive atomic effect algebras was given in author's paper [65]. Moreover, it is proved there that every weakly orthocomplete Archimedean effect algebra is orthoatomistic. This generalizes results of Ovchinnikov [38] (stated for weakly orthocomplete atomic orthomodular posets), Foulis and Bennett [11] (stated for chain-finite effect algebras), and Riečanová [50] (stated for lattice Archimedean atomic effect algebras).

8 Distributivity and associativity

Boolean algebras are distributive, i.e., $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. The orthomodular law is a special case of the distributivity using the equation for $b \leq a$ and $c = b'$. In general, suprema and infima need not exist, especially if we consider infinite sets. A result for Boolean orthoposets has been already mentioned in Section 3.

The distributivity-like properties of suprema and infima (possibly infinite) with respect to partial operations $\oplus$ and $\ominus$ and vice versa were studied by Bennett and Foulis [1] in the context of effect algebras (sometime assuming that they form a lattice) and by Chovanec and Kôpka [6] in the context of D-posets (for two-element sets assuming that the D-posets form a lattice). A unified overview of generalizations of these results is presented in author's paper [70].

A "large associativity" (also for infinite number of elements) of the partial operation $\oplus$ was studied by Riečanová [49] in the context of abelian RI-posets for complete lattices and by Ji [23] for orthocomplete effect algebras. The already mentioned paper [70] presents a generalization of these results for effect algebras.

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Education

1984 Graduated with honours from Mathematical Analysis studies of Faculty of Mathematics and Physics, Charles University, Prague. Thesis: Porous, σ-porous and other exceptional sets.

1984 Doctor of Natural Sciences (RNDr.) in Mathematical Analysis, Charles University, Prague.

1989 Diploma University Pedagogue, Institute of Technical Studies, Prague. Thesis: Mathematical part of SAT and its results for students of Faculty of Electrical Engineering of Czech Technical University in Prague.

1991 Candidate of Sciences (CSc., approx. Ph.D.) in Mathematical Analysis, Czech Technical University in Prague. Thesis: Finitely additive measures on orthostructures.

1998 Habilitation in Applied Mathematics. Czech Technical University in Prague. Thesis: Axiomatic models of quantum systems.

Occupation

1981–1984 Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Research Assistant.

1985–1987 Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Graduate Research Associate.

1987–1998 Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Assistant Professor.

1999– Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Associate Professor.

Research

Resent research interests: quantum structures (orthomodular posets, effect algebras) and measures on them.

- 39 research papers in reviewed journals (26 at journals with impact factor).
- Active participation on 24 international conferences.
- 32 research stays (Wien, Tampere, Patras, Bratislava, Budapest, Hamilton, Linz, Greifswald, Sankt Peterburg).
- Recipient of grants:
  - Union of Czech Mathematicians and Physicists (1987).
  - Czech Literary Fund (1992).
  - German Academy of Sciences (1994–1995).
  - Quantum Systems, AKTION Austria – Czech Republic (1995–1996).
  - State spaces of quantum logics, Czech Technical University in Prague (1997).
  - Representations of orthomodular structures, Ministry of Education (1998).
  - International Quantum Structures Association (2004).
  - Quantum Structures, Czech Technical University in Prague (2016).
- Participation on other 13 grants and projects (Grant Agency of the Czech Republic, Ministry of Education, Czech Technical University in Prague).

Other activities

1977–	Member of the Union of Czech Mathematicians and Physicists.
1988–	Reviewer for various mathematical journals.
1991–	Reviewer for Zentralblatt für Mathematik and Mathematical Reviews.
1992–	Member of International Quantum Structures Association.
1993–2010	Member of American Mathematical Society.
1993	Member of the organizing committee of the conference Quantum Logics Today, Prague (Czech Republic).
1994	Member of the organizing committee of the conference Quantum Structures '94, Prague (Czech Republic).
2008–2012	Member of the Council of International Quantum Structures Association.
2010	Member of the Scientific Committee of the conference Quantum Structures 2010 (Boston, USA)
2012–	Member of the Editorial Board of Mathematica Aeterna.
2012	Member of the Scientific Committee of the conference Quantum Structures 2012 (Cagliari, Italy)

List of publications

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