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Kořeny kvaternionových polynomů

Zeros of quaternionic polynomials

Summary

Let $\mathbb{H}$ be the skew field of quaternions. We will treat two types of quaternionic polynomials. Polynomials with quaternionic coefficients of the simple (one-sided) type of degree n having the form

$$p_n(z) := \sum_{j=0}^{n} a_j z^j, \quad z, a_j \in \mathbb{H}, \quad a_0,\ a_n \neq 0\,.$$

And quaternionic polynomials of the two-sided type which are, due to non-commutativity of quaternions, more complicated:

$$p_n(z) := \sum_{j=0}^{n} a_j z^j b_j, \quad z,\ a_j,\ b_j \in \mathbb{H},\ a_0 b_0 \neq 0,\ a_n b_n \neq 0.$$

The set of zeros of the simple quaternionic polynomial will separate into two classes: Let z_0 be a zero of a simple quaternionic polynomial p_n. If z_0 is not real and has the property, that $p_n(z) = 0$ for all $z \in [z_0]$, where $[z_0] := \{z \in \mathbb{H} : \Re z = \Re z_0,\ |z| = |z_0|\}$, then we will say that z_0 is a spherical zero. The set $[z_0]$ is the set of all quaternions which are similar to z_0 in the matrix sense, i. e. $[z_0] = \{z : z = hz_0h^{-1}$ for all $h \in \mathbb{H}\backslash\{0\}\}$. If z_0 is real or does not generate a spherical zero it will be called an isolated zero. Under the assumption that z_0 is a zero of p_n, either all elements in $[z_0]$ are zeros or only z_0 is a zero. In general, all complex (nonreal) zeros of simple polynomials with real coefficients are spherical zeros. And real zeros of any simple polynomial will always be isolated zeros. All zeros of the simple quaternionic polynomial p can be found very easily by finding the zeros of a corresponding real polynomial. See Janovská, Opfer, 2010, [15].

For quaternionic polynomials p of the two-sided type, we show that there are three more classes of zeros defined by the rank of a certain real 4×4 matrix $\mathbf{A}$. If a zero z_0 in one class has been found, we are able to find all zeros in the same class. The essential tool is the description of the polynomial p by a matrix equation $P(z) := \mathbf{A}(z)\, z + B(z)$, where $\mathbf{A}(z)$ is a real 4×4 matrix determined by the coefficients of the given polynomial p and $P,\ z,\ B$ are real column vectors with 4 rows. It turned out that Newton's method applied to this matrix equation represents a very effective tool in finding the zeros. See Janovská, Opfer, 2010, [16].

Some examples and applications for both cases are presented. Since quaternions may be represented isomorphically by real 4×4 matrices, the above polynomials could also be regarded as special matrix polynomials.

Souhrn

Budeme se zabývat dvěma typy kvaternionových polynomů, tzv. jednostrannými polynomy, v nichž jsou mocniny "naznámého kvaternionu" násobeny koeficienty (kvaterniony) zleva nebo zprava,

$$p_n(z) := \sum_{j=0}^{n} a_j z^j, \quad z, a_j \in \mathbb{H}, \quad a_0,\ a_n \neq 0\,,$$

a tzv. oboustrannými kvaternionovými polynomy, kde je neznámá vynásobena (různými) kvaternionovými koeficienty zleva i zprava,

$$p_n(z) := \sum_{j=0}^{n} a_j z^j b_j, \quad z,\ a_j,\ b_j \in \mathbb{H},\ a_0 b_0 \neq 0,\ a_n b_n \neq 0.$$

Množina kořenů jednostranných kvaternionových polynomů se skládá ze dvou tříd. Není-li kořen z_0 jednostranného polynomu p_n reálný a má vlastnost, že $p_n(z) = 0$ pro všechna z ze třídy ekvivalence $[z_0]$, kde $[z_0] := \{z \in \mathbb{H} : \Re z = \Re z_0,\ |z| = |z_0|\}$,, říkáme, že z_0 je sférický kořen polynomu p_n. Třída ekvivalence $[z_0]$ je množina všech kvaternionů, které jsou podobné kvaternionu z_0 v maticovém smyslu, t.j. $[z_0] = \{z : z = h z_0 h^{-1}$ pro všechna $h \in \mathbb{H}\backslash\{0\}\}$. Je-li z_0 reálné nebo negeneruje-li sférický kořen, říkáme, že z_0 je izolovaný kořen polynomu p_n.
Tedy je-li z_0 kořenem kvaternionového polynomu p_n, pak buď všechny prvky v $[z_0]$ jsou kořeny $p_n(z)$ nebo je z_0 jediný kořen v této třídě ekvivalence. Obecně jsou všechny komplexní (nereálné) kořeny jednostranných kvaternionových polynomů sférické kořeny a všechny reálné kořeny jednostranného polynomu jsou kořeny izolované. Všechny kořeny jednostranného kvaternionového polynomu p_n lze najít jako kořeny jistého přiřazeného reálného polynomu stupně $2n$, viz. Janovská, Opfer, 2010, [15] .
Oboustranné kvaternionové polynomy mohou mít ještě další tři třídy kořenů. Tyto třídy jsou definovány pomocí hodnosti jisté reálné matice $\mathbf{A} \in \mathbb{R}^{4\times 4}$. Jestliže v nějaké třídě ekvivalence najdeme kořen z_0, pak umíme najít všechny kořeny, které leží ve třídě ekvivalence $[z_0]$. Užitečným nástrojem je popis polynomu $p(z)$ pomocí maticové rovnice $P(z) := \mathbf{A}(z)\, z + B(z)$, kde matice $\mathbf{A}(z)$ je reálná matice typu 4×4 definovaná koeficienty daného polynomu p a $P,\ z,\ B$ jsou reálné sloupcové vektory o čtyřech složkách. Ukázalo se, že počítáme-li kořeny oboustranného kvaternionového polynomu Newtonovou metodou, metoda dobře konverguje, viz [16] .
Uvedeme příklady obou typů polynomů a aplikaci uvedené teorie na řešení Sylvesterovy rovnice. Poznamenejme ještě, že, protože existuje izomorfismus mezi kvaterniony a komplexními maticemi typu 2×2 a také mezi kvaterniony a jistými reálnými maticemi typu 4×4, lze kvaternionové polynomy považovat za speciální maticové polynomy.

Klíčová slova: Kořeny jednostranných kvaternionových polynomů, kořeny oboustranných kvaternionových polynomů, klasifikace kořenů, izolované kořeny, sférické kořeny, Sylvesterova rovnice pro kvaterniony.

Keywords: Zeros of simple quaternionic polynomials, zeros of two-sided quaternionic polynomials, classification of zeros, izolated zeros, spherical zeros, Sylvester's equation in quaternions.

Contents

1 Introduction

Quaternions are a very useful tool for describing motions of rigid bodies. If a chair is thrown into the air, then its motion can be described by the use of quaternions in an economic fashion. Thus, computer games which involve many such motions, are a preferred field of applications of quaternionic algebra. The same is true for the construction of industrially produced robots. One can find more applications by employing the internet. However, all these applications are essentially based on the capability of the multiplication with a single quaternion in the sense of an orthogonal transformation. More complex (in the sense of complicated) structures like matrices or polynomials defined by quaternions are mainly studied from a theoretical point of view. There are few isolated older papers treating quaternionic problems numerically. Two papers on eigenvalue problems are by Dongarra, Gabriel, Koelling, and Wilkinson, 1984, [3, 4], there is also a paper on the QR decomposition of quaternionic matrices by Bunse-Gerstner, Byers, and Mehrmann, 1989, [2].

The first attempts to find the zeros of a quaternonic polynomial were made by Niven in 1941; so called simple quaternionic polynomials were considered. The idea of Niven was to divide the polynomial by a quadratic polynomial with (certain) real coefficients and to adjust the coefficients of the quadratic polynomial by an iterative procedure in such a way that the remainder of the division vanished. Finally, it was shown, that the set of zeros of the resulting quadratic polynomial also contained quaternions. The first numerically working algorithm based on these ideas was presented by Serôdio, Pereira, and Vitório in 2001, [35]. Further contributions to polynomials with quaternionic coefficients were made by Pumplün and Walcher, 2002, [34], de Leo, Ducati, and Leonhard, 2006, [29], Gentile and Struppa, 2007, [7], Gentile, Struppa, and Vlacci, 2008, [8]. There is a very useful overview on quaternionic matrix problems by Zhang,1997, [40]. A well working procedure for finding all zeros of simple quaternionic polynomials can be found in Janovská, Opfer, 2010, [15], classification of zeros of two–sided quaternionic polynomials is given in Janovská, Opfer, 2010, [16].

Let us note that linear mappings in the space of quaternions are not necessarily reduced to the form $\ell = ax$, they may also have the form $\ell(x) = axb$ and even $\ell(x) = axb + cxd + \cdots + yxz$ with a finite, but arbitrary number of terms. In the paper by Janovská, Opfer, 2008, [19], we have shown how to treat such problems. However, there is still a question which algorithms can be applied to solving quaternionic linear problems. The answer is strongly related to matrix decompositions. In Opfer, 2005, [32], the CG–algorithm was applied.

Polynomials with quaternionic coefficients located on only one side of the powers (we call them simple polynomials) may have two different types of zeros: isolated and spherical zeros. We will give a characterization of the types of the zeros and, based on this characterization, we will present an algorithm for producing all zeros including their types without using an iteration process which requires convergence. The main tool is the representation of the powers of a quaternion as a real, linear combination of the quaternion and the number one (as introduced by Pogorui and Shapiro, 2004, [33]), and the use of a real companion polynomial which already was introduced for the first time by Niven, 1941, [30].

Let us consider quaternionic polynomials whose coefficients are located at both sides of the powers (we call them two–sided polynomials). We show that in this case there are, in addition, three more classes of zeros defined by the rank of a certain real 4×4 matrix. This information can be used to find all zeros in the same class if only one zero in that class is known. The essential tool is the description of the polynomial p by a matrix equation $P(z) := \mathbf{A}(z)z + B(z)$, where $\mathbf{A}(z)$ is a real 4×4 matrix determined by the coefficients of the given polynomial p and P, z, B are real column vectors with four rows. This representation allows also to include two–sided polynomials which contain several terms of the same degree.

We applied Newton's method to $P(z) = 0$ and at least for isolated zeros this method turned out to be a very effective tool in finding those zeros. It allows us also to prove that the number of zeros of a quaternionic, two-sided polynomial p is not bounded by the degree of that polynomial.

We conjecture that the bound is $2n$. The paper on Newton's method, 2007, [23], shows that it is also possible to apply this very important and powerful method to quaternionic cases without loss of approximation power.

2 Preliminaries

Let us introduce some notation. By $\mathbb{R}$, $\mathbb{C}$ we denote the fields of real and complex numbers, respectively, and by $\mathbb{Z}$ the set of integers. By $\mathbb{H}$ (in honor of the founder, Hamilton, 1843) we denote the skew field of quaternions.

Let $\mathbb{H} = \mathbb{R}^4$ be equipped with the ordinary vector space structure with an additional multiplicative operation $\mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ which most easily can be defined by a multiplication of the four basis elements

$$(1,0,0,0) = \mathbf{1}, \quad (0,1,0,0) = \mathbf{i}, \quad (0,0,1,0) = \mathbf{j}, \quad (0,0,0,1) = \mathbf{k} \ :$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}\,. \tag{1}$$

An element $x = (x_1, x_2, x_3, x_4) \in \mathbb{H}$ has the representation

$$x = \ x_1\mathbf{1} + x_2\mathbf{i} + x_3\,\mathbf{j} + x_4\mathbf{k}, \tag{2}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$, $\Re\, x = x_1$ is the real part of x, $\operatorname{Vec} x = x_2\mathbf{i} + x_3\,\mathbf{j} + x_4\mathbf{k}$ is the vector part of x. We will identify the quaternion $x = (x_1, 0, 0, 0)$ with the real number x_1, the quaternion $x = (x_1, x_2, 0, 0)$ will be identified with the complex number $x_1 + \mathrm{i}x_2$. If we denote $\mathbf{v} = (x_2, x_3, x_4) \in \mathbb{R}^3$ the vector part of x then, the quaternion x has the representation:

$$x = (x_1, \mathbf{v}), \quad x_1 \in \mathbb{R}, \quad \mathbf{v} \in \mathbb{R}^3. \tag{3}$$

For $x = (x_1, x_2, x_3, x_4) = (x_1, \mathbf{v}) \in \mathbb{H},\ y = (y_1, y_2, y_3, y_4) = (y_1, \mathbf{w}) \in \mathbb{H}$ it follows from (1) that

$$\begin{aligned} x\,y \ &= \quad (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)\,\mathbf{1} + (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3)\,\mathbf{i} \\ &\quad + (x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2)\,\mathbf{j} + (x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1)\,\mathbf{k} \\ &= \ (x_1y_1 - \mathbf{v}\cdot\mathbf{w},\ x_1\mathbf{w} + y_1\mathbf{v} + \mathbf{v}\times\mathbf{w}), \end{aligned} \tag{4}$$

where $\cdot$, $\times$ are the scalar, vector products in $\mathbb{R}^3$, respectively. Obviously, in general, the multiplication is not commutative. Given x according to (2), the conjugate $\overline{x}$ of x is defined to be

$$\overline{x} = (x_1, -x_2, -x_3, -x_4). = \Re\, x - \operatorname{Vec} x. \tag{5}$$

We define the absolute value of x by

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}. \tag{6}$$

The space $\mathbb{H}$ is a normed vector space over $\mathbb{H}$, where the norm is introduced in (6).

Let $x = (x_1, x_2, x_3, x_4),\ y = (y_1, y_2, y_3, y_4) \in \mathbb{H}$ be two quaternions, $r \in \mathbb{R}$. Then,

$$\begin{aligned} x^2 &= x_1^2 - x_2^2 - x_3^2 - x_4^2 + 2x_1(x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = 2(\Re x)x - |x|^2, \\ (\operatorname{Vec} x)^2 &= -x_2^2 - x_3^2 - x_4^2 = -|\operatorname{Vec} x|^2, \\ \Re(xy) &= x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 = \Re(yx); && (7) \\ \overline{xy} &= \overline{y}\,\overline{x}, \qquad\qquad \overline{\overline{x}} = x; && (8) \\ |x|^2 &= x\,\overline{x} = \overline{x}\,x, \qquad |xy| = |yx| = |x||y|; && (9) \\ x^{-1} &= \frac{\overline{x}}{|x|^2} \quad \text{for} \quad x \in \mathbb{H}\backslash\{0\}. && (10) \end{aligned}$$

We shall use the following notation:

$$\operatorname{sgn} x = \frac{\overline{x}}{|x|} \text{ for } x \in \mathbb{H}\backslash\{0\}. \tag{11}$$

It has the property that $(\operatorname{sgn} x)\, x = x \operatorname{sgn} x = |x|$.

Let us see a small example. Let $p_2(z) = z^2 + 1$. This quadratic polynomial has no real zero and it has two imaginary zeros $z_{1,2} = \pm\mathbf{i}$. How many zeros it has as a quadratic quaternionic polynomial? Let $z = h^{-1} z_{1,2} h$, where $h \in \mathbb{H} \setminus \{0\}$ is arbitrary. Then

$$z^2 + 1 = h^{-1} z_{1,2} h\, h^{-1} z_{1,2} h + 1 = h^{-1} \mathbf{i}^2 h + 1 = 0\,.$$

As a quadratic quaternionic polynomial, p_2 has infinitely many zeros.

Definition 1. Two quaternions $a, b \in \mathbb{H}$ are called equivalent, denoted by $a \sim b$, if

$$a \sim b \quad \Longleftrightarrow \quad \exists\, h \in \mathbb{H} \setminus \{0\} \quad \text{such that} \quad a = h^{-1} b h. \tag{12}$$

The set

$$[a] := \big\{ u \in \mathbb{H} \,:\, u = h^{-1} a h \quad \text{for all} \quad h \in \mathbb{H} \setminus \{0\} \big\} \tag{13}$$

will be called an equivalence class of a.

The relation $\sim$ is indeed an equivalence relation. Equivalent quaternions a, b can be easily recognized by

$$a \sim b \quad \Longleftrightarrow \quad \Re a = \Re b \quad \text{and} \quad |a| = |b|, \quad (\text{see}[20]). \tag{14}$$

Let a_1 be real. Then $[a_1] = \{a_1\}$, which means, that in this case, the equivalence class consists only of one element, $\{a_1\}$. If a is not real, then $[a]$ always contains infinitely many elements which due to (12), (13), and (14), can be characterized by

$$[a] = \{ z \in \mathbb{H} \,:\, \Re z = \Re a, \quad \text{and} \quad |z| = |a| \}\,, \tag{15}$$

and the equivalence class $[a]$ can be regarded as a two dimensional sphere in $\mathbb{R}^4$.

Let $z := (z_1, z_2, z_3, z_4) \in \mathbb{H}$. Then it follows from (15) that $\overline{z} \in [z]$. If $z \in \mathbb{H}$ will not be real then the equivalence class $[z]$ contains exactly two complex numbers $a \in \mathbb{C}$ and $\overline{a} \in \mathbb{C}$ where

$$a = (z_1, +\sqrt{z_2^2 + z_3^2 + z_4^2}, 0, 0) = z_1 + |\text{Vec z}|\mathbf{i} \in [\text{z}],$$

i.e., a is the only complex element in $[z]$ with a non negative imaginary part. The complex number a will be called the complex representative of $[z]$.

There is an isomophism between the field of quaternions $\mathbb{H}$ and a certain class of matrices in $\mathbb{C}^{2\times 2}$. Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$. Let us put $\alpha = a_1 + a_2\mathbf{i},\ \beta = a_3 + a_4\mathbf{i}$. Then the set of matrices of the form

$$\widetilde{\mathbf{H}} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with ordinary matrix addition and multiplication is isomorphic to $\mathbb{H}$, see B. L. van der Waerden, 1960 (1st ed. 1936), [37].

This isomorphism is very useful in quantum mechanics. In modelling of electronic structures of molecules and solids containing heavy atoms the use of relativistic kinematics is required namely, when effects that modify symmetry (spin-orbit coupling) are concerned.

This leads to complex systems of equations with matrices in which a scalar element $a \in \mathbb{R}$ is replaced by $\widetilde{\mathbf{H}}$, i.e. by a 2×2 matrix with complex elements. Due to the isomorphism it means that we can work with quaternionic matrices. It has some advantages (increased accuracy, economy of storage), but on the other hand it needs more computational effort.

There is also another isomorphic matrix representation of quaternions.
We introduce two mappings $\omega_1, \omega_2 : \mathbb{H} \longrightarrow \mathbb{R}^{4\times 4}$ by

$$\omega_1(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \tag{16}$$

$$\omega_2(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4\times 4}. \tag{17}$$

The first mapping ω_1 represents the isomorphic image of a quaternion $a = (a_1, a_2, a_3, a_4)$ in the matrix space $\mathbb{R}^{4\times 4}$. Thus we have

$$\omega_1(ab) = \omega_1(a)\omega_1(b).$$

The two matrices $\omega_1(a), \omega_2(b)$ coincide if and only if $a = b \in \mathbb{R}$, see Gürlebeck, Sprössig, 1995, [11].

The second mapping ω_2 introduced by Aramanovitch, 1995, [1], has the important property that it reverses the multiplication order

$$\omega_2(ab) = \omega_2(b)\omega_2(a).$$

From the definition (16) it follows that

$$\omega_1(a)^{\mathrm{T}} = \omega_1(\overline{a}), \qquad \omega_2(b)^{\mathrm{T}} = \omega_1(\overline{b}).$$

where the superscript $^{\mathrm{T}}$ denotes transposition. It follows, that both matrices are orthogonal in the sense $\omega_1(a)\omega_1(a)^{\mathrm{T}} = \omega_1(a)\omega_1(\overline{a}) = |a|^2\,\mathbf{I}$, $\omega_2(b)\omega_2(b)^{\mathrm{T}} = |b|^2\,\mathbf{I}$, where $\mathbf{I}$ is the (4×4) identity matrix.

Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}$. We introduce an column operator $\mathrm{col} : \mathbb{H} \longrightarrow \mathbb{R}^4$ by

$$\mathrm{col}(a) := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}. \tag{18}$$

This column operator enables us to regard a quaternion as a matrix with one column and four rows. It is linear over $\mathbb{R}$, i. e.

$$\mathrm{col}(\alpha a + \beta b) = \alpha \mathrm{col}(a) + \beta \mathrm{col}(b), \quad a, b \in \mathbb{H},\ \alpha, \beta \in \mathbb{R}. \tag{19}$$

Lemma 1. (Aramanovitch, 1995, [1], Gürlebeck, Sprössig, 1995, [11]) For arbitrary quaternions a, b, c we have

$$\begin{aligned} \mathrm{col(ab)} &= \omega_1(a)\mathrm{col(b)} = \omega_2(\mathrm{b})\mathrm{col(a)}, \\ \mathrm{col(abc)} &= \omega_1(a)\omega_2(c)\mathrm{col(b)}, \\ \mathrm{col(abc)} &= \omega_1(a)\omega_1(b)\mathrm{col(c)} = \omega_2(\mathrm{c})\omega_2(\mathrm{b})\mathrm{col(a)}. \end{aligned}$$

For more properties of these mappings, see [16].

Let us put

$$\omega_3(a, b) := \omega_1(a)\omega_2(b) \in \mathbb{R}^{4\times 4}, \quad a, b \in \mathbb{H}. \tag{20}$$

Lemma 2. The matrix $\omega_3(a,b)$ is normal and orthogonal in the sense

$$\omega_3(a,b)^{\rm T}\omega_3(a,b) = \omega_3(a,b)\omega_3(a,b)^{\rm T} = |a|^2|b|^2\,\mathbf{I}\,.$$

Thus, all eigenvalues of $\omega_3(a,b)$ have the same absolute value $|a||b|$.

For proof, see [16] .

Let $\mathbf{A}$ be a square matrix over $\mathbb{C}$ of order n. Then, see e.g. Horn & Johnson, [13], any power $\mathbf{A}^j$ belongs to a linear hull of the powers of the matrix $\mathbf{A}$ up to the degree of the minimal polynomial:

$$\mathbf{A}^j \in \langle \mathbf{I},\, \mathbf{A},\, \mathbf{A}^2,\, \dots\, , \mathbf{A}^{\nu-1}\rangle\,,\; j\in\mathbb{N}\,,$$

where ν is the degree of the minimal polynomial of $\mathbf{A}$. In general, $\nu \le n$ and ν divides n, see e.g. Horn & Johnson, [13]. In particular for $\nu = 2$ we have

$$\mathbf{A}^j = \alpha_j\,\mathbf{I} + \beta_j\mathbf{A}\,,\quad j\in\mathbb{N}\,,$$

and the coefficients $\alpha_j,\ \beta_j$ can be computed by recursion.

We will apply this theory to quaternions or, more precisely, to the real matrix $\omega_1(a)$ that represents the quaternion a. It has the minimal polynomial

$$\mu(\omega(a)) = \lambda^2 - 2\lambda a_1 + |a|^2 \quad \text{i.e.} \quad \nu = 2\,.$$

As a consequence, all powers $z^j, j\in\mathbb{Z}$ of a quaternion z have the form $z^j = \alpha z + \beta$ with real α, β. In particular,

$$z^2 = 2\Re z\, z - |z|^2. \tag{21}$$

In order to determine the numbers α, β we set up the following iteration (for negative j and non vanishing z we use $z^{-1} = \dfrac{\overline{z}}{|z|^2}$ instead of z)

$$\begin{aligned} z^j &= \alpha_j z + \beta_j, \quad \alpha_j, \beta_j \in \mathbb{R}, \quad j = 0,1,\dots, \text{ where} && (22)\\ \alpha_0 &= 0, \quad \beta_0 = 1, && (23)\\ \alpha_{j+1} &= 2\Re z\,\alpha_j + \beta_j, && (24)\\ \beta_{j+1} &= -|z|^2\alpha_j, \quad j = 0,1,\dots && (25)\end{aligned}$$

The corresponding iteration given by Pogorui and Shapiro, 2004, [33], is a three term recursion whereas this one (formulas (23) to (25) is a two term recursion. Thus, they differ, formally. In some cases two term recursions are more stable, than the corresponding three term recursion. For an example, see Laurie, 1999, [28]. The given recursion is a very economic means to calculate the powers of a quaternion. In order to compute all powers of $z \in \mathbb{H}$ up to degree n, one needs $n-1$ quaternionic multiplications, where one quaternionic multiplication (see (4)) needs 28 flops real floating point operations, whereas the recursion (23) to (25) only needs $3n$ flops. The sequence $\{\alpha_j\}$ is defined by a difference equation of order two with constant coefficients. Using the theory of difference equations, it is possible to give a closed form solution for α_j. There are two versions valid for the case $z \notin \mathbb{R}$. One of the versions is purely real, the other is formally complex. The real version of the solution is as follows:

$$\alpha_j = \frac{\Im\{u_1^j\}}{\sqrt{|z|^2-(\Re z)^2}}, \quad u_1 := \Re z + \mathbf{i}\sqrt{|z|^2-(\Re z)^2}, \quad \sqrt{|z|^2-(\Re z)^2} > 0,\ j\ge 0, \tag{26}$$

where u_1 is one of the two complex solutions of $u^2 - 2\Re z\, u + |z|^2 = 0$. Formula (26) for α_j is of course easier to program than the iteration (23) to (25). However, since a power is involved, an economic use of (26) would also require an iteration.

3 Simple (one–sided) quaternionic polynomials

Let $p_n(z)$ be a given polynomial of degree n, n positive integer,

$$p_n(z) = \sum_{j=0}^{n} a_j z^j\,, \quad z,\, a_j \in \mathbb{H},\ j = 0, 1, 2, \ldots, n, \quad a_0,\ a_n \neq 0\,. \tag{27}$$

Polynomial $p_n(z)$ in (27) is called one-sided (or simple) quaternionic polynomial.

Let us remark that the assumption $a_0 \neq 0$ guarantees that the origin is never a zero of p_n, the assumption $a_n \neq 0$ ensures that the degree of the polynomial is not less then n. Without loss of generality we could assume $a_n = 1$. It should be noted that the general form of a quaternionic monomial would be $a_0 \cdot z \cdot a_1 \cdot z \cdot a_2 \cdots a_{j-1} \cdot z \cdot a_j$, i.e., the above p_n is only a very special type of quaternionic polynomial. See [31] for some statements on polynomials of general type. It also should be noted that it is still possible to evaluate $p_n(z)$ by Horners scheme, although coefficients and argument are in $\mathbb{H}$.

The set of zeros of a polynomial of type (27) will separate into two classes.

Definition 2. Let z_0 be a zero of a simple quaternionic polynomial (27). If z_0 is not real and has the property that $p_n(z) = 0$ for all $z \in [z_0]$, then we will say that z_0 is (or generates) a spherical zero. If z_0 is real or does not generate a spherical zero, it is called an isolated zero. The number of zeros of p_n will be defined as the number of equivalence classes, which contain at least one zero of p_n.

In what follows, we will see that under the assumption that z_0 is a zero of p_n, either all elements in $[z_0]$ are zeros, or z_0 is the only zero in $[z_0]$.

By means of (22) the polynomial p_n can be written as

$$p_n(z) := \sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} a_j(\alpha_j\, z + \beta_j) = \left(\sum_{j=0}^{n} \alpha_j a_j\right) z + \sum_{j=0}^{n} \beta_j a_j =: A(z)z + B(z). \tag{28}$$

Theorem 1. Let $z_0 \in \mathbb{H}$ be fixed. Then $A(z) = \mathrm{const},\ \mathrm{B(z)} = \mathrm{const}$ for all $z \in [z_0]$, where A, B are defined in (28). Let z_0 be a zero of p_n. Then,

$$p_n(z_0) = A(z)z_0 + B(z) = 0 \quad \text{for all } z \in [z_0]. \tag{29}$$

The quantities A, B in (29) can only vanish simultaneously. If $A(z_0) = 0$ and if z_0 is not real, then, z_0 generates a spherical zero of p_n. If $A(z_0) \neq 0$, then z_0 is an isolated zero.

Proof. From (23) to (25) it is clear, that the coefficients $\alpha_j,\ \beta_j,\ j \geq 0$, are the same for all z with the same $\Re z, |z|$. Thus, the coefficients are the same for all $z \in [z_0]$, therefore, $A(z) = \mathrm{const},\ \mathrm{B(z)} = \mathrm{const}$ for all $z \in [z_0]$. If $A(z_0) = 0$, then necessarily $B(z_0) = 0$, and vice versa. And $p(z) = 0$ for all $z \in [z_0]$. This implies that z_0 generates a spherical zero if z_0 is not real. Recall, that $z_0 \neq 0$. Let $A(z_0) \neq 0$ and z_0 not isolated. This case leads to a contradiction as shown in the next theorem. ☐

Theorem 2. Let $z_0, z_1 \in \mathbb{H}$ be two different zeros of p_n with $z_0 \in [z_1]$. Then $p_n(z) = 0$ for all $z \in [z_1]$ and z_0 generates a spherical zero of p_n and $A(z) = B(z) = 0$ for all $z \in [z_0]$, where A, B are defined in (28). In particular, z_0 is a spherical zero of p_n if and only if $A(z_0) = 0$, provided, z_0 is not real.

Proof. Since z_0, z_1 are assumed to be different and to belong to the same equivalence class, they cannot be real. From (28) it follows that $p_n(z_j) = A(z)z_j + B(z) = 0$ for all $z \in [z_0] = [z_1],\ j = 0, 1$. Taking differences, we obtain $p_n(z_0) - p_n(z_1) = A(z)(z_0 - z_1) = 0$ for all $z \in [z_1] = [z_0]$, implying $A(z) = 0$. According to Theorem 1, the zero z_0 generates a spherical zero of p_n. If $A(z_0) \neq 0$, the zero, z_0, cannot be spherical. See also Pogorui and Shapiro, [33]. ☐

Thus, we have the following classification of the zeros z_0 of p_n given in (27):

1. z_0 is real. By definition, z_0 is isolated.
2. z_0 is not real.
   - $A(z_0) = 0 \ \Rightarrow z_0$ is spherical, all $z \in [z_0]$ are zeros of p_n.
   - $A(z_0) \neq 0 \ \Rightarrow z_0$ is isolated.

The computation of all zeros of p_n, including their types, can be reduced to the computation of all zeros of a real polynomial of degree $2n$.

Let p_n be the polynomial defined in (27) with the quaternionic coefficients $a_0, a_1, \ldots, a_n$. Following Niven, 1941, [30], or more recently Pogorui and Shapiro, 2004, [33], we define the polynomial q_{2n} of degree $2n$ with real coefficients by

$$q_{2n}(z) \quad := \quad \sum_{j,k=0}^{n} \overline{a_j} a_k z^{j+k} = \sum_{k=0}^{2n} b_k z^k, \quad z \in \mathbb{C}, \quad \text{where} \tag{30}$$

$$b_k \quad := \quad \sum_{j=\max(0,k-n)}^{\min(k,n)} \overline{a_j} a_{k-j} \in \mathbb{R}, \quad k = 0, 1, \ldots, 2n. \tag{31}$$

We will call q_{2n} the companion polynomial of the quaternionic polynomial p_n. Since it has real coefficients, we may assume that it is always possible to find all (real and complex) zeros of q_{2n}. How are the quaternionic zeros of p_n related to the real or complex zeros of q_{2n}? Here is the answer. For detailed proof, see [15].

Theorem 3. Let p_n be a given simple quaternionic polynomial and let q_{2n} be the corresponding companion polynomial. Then

1. Let $z_0 \in \mathbb{R}$. Then, $q_{2n}(z_0) = 0 \iff p_n(z_0) = 0$. The set of the real zeros is the same for p_n and for q_{2n}.
2. Let z_0 be a nonreal zero of q_{2n} and let $A(z_0) = 0$. See (28) for the definition of A. Then, z_0 generates a spherical zero of p_n.
3. Let x is a nonreal, complex zero of q_{2n} with the property that $A(x) \neq 0$. Then, z

   $$z := -A(x)^{-1} B(x) = -\frac{\overline{A(x)} B(x)}{|A(x)|^2}. \tag{32}$$

   is an isolated zero of p_n. If we use the notation

   $$x = (x_1, x_2, 0, 0); \quad z := (z_1, z_2, z_3, z_4); \quad \overline{A}B := (v_1, v_2, v_3, v_4). \tag{33}$$

   and $|v| = \sqrt{v_2^2 + v_3^2 + v_4^2}$ we can give z also the following form

   $$Z := (x_1, -\frac{|x_2|}{|v|} v_2, -\frac{|x_2|}{|v|} v_3, -\frac{|x_2|}{|v|} v_4). \tag{34}$$

There is still one missing link. Is it true, that the zeros of the companion polynomial q_{2n} really exhaust all zeros of p_n or is it possible that p_n has a zero which we do not find by checking all zeros of q_{2n}?

Theorem 4. Let $p_n(z) = 0$ where p_n is defined in (27). Then, there is an $x \in \mathbb{C}$ with $x \in [z]$ such that $q_{2n}(x) = 0$, where q_{2n} is defined in (30), (31).

Example 1. Let

$$p_6(z) := z^6 + \mathbf{j}z^5 + \mathbf{i}z^4 - z^2 - \mathbf{j}z - \mathbf{i}. \tag{35}$$

Then, the companion polynomial for p_6 is

$$q_{12}(x) = x^{12} + x^{10} - x^8 - 2x^6 - x^4 + x^2 + 1. \tag{36}$$

The twelve zeros of q_{12} are

$$1 \text{ (twice)}, \quad -1 \text{ (twice)}, \quad \pm\mathbf{i} \text{ (twice each)}, \quad 0.5(\pm 1 \pm \mathbf{i}).$$

There are two different, real zeros, $z_{1,2} = \pm 1$ which are also zeros of p_6. There is one spherical zero, $z_3 = \mathbf{i}$, of p_6 ($-\mathbf{i}$ generates the same spherical zero). And, finally there are two isolated zeros which have to be computed from $x = 0.5\,(\pm 1 \pm \mathbf{i})$ by formula (34). This formula yields

$$z_4 := 0.5\,(1, -1, -1, -1), \quad z_5 := 0.5\,(-1, 1, -1, -1).$$

So far our simple quaternionic polynomial had coefficients on the left side of the powers. Let

$$\tilde{p}_n(z) := \sum_{j=0}^{n} z^j a_j, \quad z, a_j \in \mathbb{H}, \quad j = 0, 1, 2, \ldots, n, \quad a_0,\, a_n \neq 0\,. \tag{37}$$

be a given polynomial with coefficients on the right side of the powers.

We apply the former theory to

$$p_n(z) := \overline{\tilde{p}_n(\overline{z})} = \sum_{j=0}^{n} \overline{a_j} z^j, \quad z, a_j \in \mathbb{H}, \quad j = 0, 1, 2, \ldots, n, \quad a_0,\, a_n \neq 0. \tag{38}$$

Lemma 3. The two polynomials

$$\tilde{p}_n(z) := \sum_{j=0}^{n} z^j a_j \quad \text{and} \quad p_n(z) := \sum_{j=0}^{n} \overline{a_j} z^j$$

have the same real and spherical zeros.

Let us summarize the algorithm for finding zeros of a simple quaternionic polynomial (27),

$$p_n(z) := \sum_{j=0}^{n} a_j z^j, \quad z, a_j \in \mathbb{H}, \quad j = 0, 1, \ldots, n,\ a_n = 1\,, a_0 \neq 0,\ n \geq 1\,.$$

1. Compute the real coefficients $b_0, b_1, \ldots, b_{2n}$ of the companion polynomial q_{2n} by formula (31). Make sure that they are real.
2. Compute all $2n$ (real and complex) zeros of q_{2n}, (in MATLAB, use the command `roots`). Denote these zeros by $z_1, z_2, \ldots, z_{2n}$ and order these zeros (if necessary) such that $z_{2j-1} = \overline{z_{2j}},\ j = 1, 2, \ldots, n$. If a specific z_{2j_0-1} is real, then, it means that $z_{2j_0-1} = z_{2j_0}$.
3. Define an integer vector `ind` (like *indicator*) of length n and set all components to zero. Define a quaternionic vector Z of length n and set all components to zero.
   `For j:=1:n do` Put $z := z_{2j-1}$.
   (a) `if` z is real, $Z(j) := z$; go to the next step; `end if`
   (b) Compute $v := \overline{A(z)}B(z)$ by formula (28), with the help of (23) to (25).
   (c) `if` $v = 0$, put `ind`$(j) := 1$; $Z(j) := z$; go to the next step; `end if`

(d) `if` $v \neq 0$, let $(v_1, v_2, v_3, v_4) := v$. Compute $|w| := \sqrt{v_2^2 + v_3^3 + v_4^2}$, put

$$(34') \qquad Z(j) := \left(\Re(z), -\frac{|\Im(z)|}{|w|}v_2, -\frac{|\Im(z)|}{|w|}v_3, -\frac{|\Im(z)|}{|w|}v_4\right).$$

```
end if
end for
```

The result of this algorithm will be an integer vector `ind` and a quaternionic vector Z, both of length n. If $\mathtt{ind}(j) = 1$, it signals that the complex number $Z(j)$ generates a spherical zero of p_n. In all other cases $Z(j)$ will be an isolated zero of p_n. Though the quaternionic vector Z has length n, the number of pairwise distinct entries may be smaller.

There are two delicate decisions to make in the above algorithm. In step 3(a) one has to decide whether z is real. And in step 3(c) one has to decide whether v is zero. Since a real zero of q_{2n} is always a double zero a test of the form $|\Im(z)| < 10^{-6}$ is appropriate.

We made some hundred tests with polynomials p_n of degree $n \leq 50$ with random integer coefficients in the range $[-5, 5]$ and with real coefficients in the range $[0, 1]$. In all cases we found only (non real) isolated zeros z. The test cases showed $|p_n(z)| \approx 10^{-13}$. Real zeros and spherical zeros did not show up. If n is too large, say $n \approx 100$, then it is usually not any more possible to find all zeros of the companion polynomial by standard means (say `roots` in MATLAB) because the coefficients of the companion polynomial are too large.

Conclusion The described procedure finds all zeros of the simple quaternionic polynomial p_n. The set of zeros consists of at least one and at most n elements, where the spherical zeros of the same equivalence class count as one zero.

4 Two–sided quaternionic polynomials

Let us recall that a general quaternionic polynomial consists of a sum of terms of the type

$$t_j(z) := a_{0j} \cdot z \cdot a_{1j} \cdots a_{j-1,j} \cdot z \cdot a_{jj}\,, \quad z, a_{0j}, a_{1j}, \ldots, a_{jj} \in \mathbb{H},\ j \geq 0.$$

We call this term a monomial of degree j. Since there may be several terms of the same degree we have to enumerate the terms. We do that in the form

$$t_{jk}(z) := a_{0j}^{(k)} \cdot z \cdot a_{1j}^{(k)} \cdots a_{j-1,j}^{(k)} \cdot z \cdot a_{jj}^{(k)}\,, \quad k = 1, 2, \ldots, k_j,\ k_j \geq 0. \tag{39}$$

The case $k_j = 0$ means that there is no monomial of degree j. A general, quaternionic polynomial of degree n takes the form

$$p(z) := \sum_{j=0}^{n} \sum_{k=1}^{k_j} t_{jk}(z)\,. \tag{40}$$

There are some recent results on these polynomials in a paper by Opfer, 2009, [31]. The essential result is by Eilenberg and Niven 1944, [5]. It says that such a polynomial has at least one zero, provided the number of monomials of degree n is only one. It is clear, that also polynomials which contain terms like $a \cdot z^2 \cdot b \cdot z^4 \cdot c$ are included in the form (39). One only needs to choose some of the coefficients to be real. Let $z \in \mathbb{R}$ be a real zero of p defined in (40). Since a real z commutes with all quaternions the polynomial can be written in the form

$$p(z) = \sum_{j=0}^{n} A_j z^j \quad \text{where} \quad A_j := \sum_{k=1}^{k_j} a_{0j}^{(k)}\, a_{1j}^{(k)} \cdots a_{jj}^{(k)}\,, \quad z \in \mathbb{R}. \tag{41}$$

Example 2. Let $z \in \mathbb{R}$,

$$p(z) := z^2 + azbzc + dzc + f\,. \tag{42}$$

The polynomial (41) reads in this case

$$p(z) = (1 + abc)z^2 + dez + f\,.$$

Let us choose

$$a := \mathbf{i},\ b := \mathbf{j},\ c := -\mathbf{k},\ d := \mathbf{i} + \mathbf{j},\ e := \mathbf{j} + \mathbf{k},\ f := -1 - \mathbf{i} + \mathbf{j} - \mathbf{k}\,,$$

such that $p(z) = 2z^2 + (-1, 1, -1, 1)z + (-1, -1, 1, -1)$. The companion polynomial q in (30), (31) of degree four has 1 as a double zero and has no other real zero. Thus, the polynomial p in (42) has exactly one real zero, namely 1. If in the general case the companion polynomial q has no real zero, then also the given polynomial p in (40) has no real zero. Because of these results, we will always disregard the discussion on real zeros in the sequel.

Now, we will treat the two–sided quaternionic polynomial in the form (43). To the polynomials with multiple terms of the same degree we will return later.

$$p(z) := \sum_{j=0}^{n} a_j z^j b_j,\ z,\ a_j,\ b_j \in \mathbb{H},\ j = 0, 1, \ldots, n \in \mathbb{N}\,,\ a_0 b_0 \neq 0,\ a_n b_n \neq 0\,. \tag{43}$$

By means of (22) the two–sided quaternionic polynomial p can be written as

$$p(z) := \sum_{j=0}^{n} a_j z^j b_j = \sum_{j=0}^{n} a_j(\alpha_j\, z + \beta_j)b_j \tag{44}$$

$$= \sum_{j=0}^{n} \alpha_j a_j\, z\, b_j + \sum_{j=0}^{n} \beta_j a_j b_j = C(z) + B(z), \quad \text{where} \tag{45}$$

$$C(z) := \sum_{j=0}^{n} \alpha_j a_j\, z\, b_j, \qquad B(z) := \sum_{j=0}^{n} \beta_j a_j b_j\,. \tag{46}$$

Lemma 4. Let C be defined as in (46). Then, $C : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is a linear mapping over $\mathbb{R}$. Let z_0 be nonreal. Then, $B(z)$, defined in (46), is constant for $z \in [z_0]$. If $p(z) = 0$ for some $z \in \mathbb{H}$, then $C(z) = B(z) = 0$ or $C(z) \neq 0$ and $B(z) \neq 0$.

If we apply the column operator (18), relations (19) and Lemma 1 to the polynomial p we obtain

Theorem 5. Let $p(z) := C(z) + B(z)$ be defined as in (44) to (46). Then,

$$\mathrm{col(p(z))} = \left(\sum_{j=0}^{n} \alpha_j \omega_3(a_j, b_j)\right) \mathrm{col(z)} + \sum_{\mathrm{j=0}}^{\mathrm{n}} \beta_{\mathrm{j}} \mathrm{col(a_j, b_j)} \tag{47}$$

$$= \mathbf{A}(z)\mathrm{col(z)} + \mathrm{col(B(z))}\,, \quad \text{where} \tag{48}$$

$$\mathbf{A}(z) = \left(\sum_{j=0}^{n} \alpha_j \omega_3(a_j, b_j)\right) \in \mathbb{R}^{4\times 4}, \quad \mathrm{col(B(z))} = \sum_{\mathrm{j=0}}^{\mathrm{n}} \beta_{\mathrm{j}} \mathrm{col(a_j, b_j)}\,. \tag{49}$$

Lemma 5. Let z_0 be nonreal. Then, the matrix $\mathbf{A}(z)$, defined in Theorem 5 is constant for $z \in [z_0]$.

Instead of considering the equation $p(z) = 0$ we consider the equivalent equation

$$P(z) := \mathrm{col(p(z))} = \mathbf{A}\mathrm{(z)col(z)} + \mathrm{col(B(z))} = \mathrm{col(0)} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{50}$$

Theorem 6. Let z be a nonreal zero of p such that equation (50) is valid. Then, this equation remains valid if in $\mathbf{A}(z),\ B(z)$ the zero z is replaced with the complex representative z_0 of $[z]$.

From these results we obtain a classification of the zeros of two-sided quaternionic polynomial p as follows:

Definition 3. Let z be a zero of p, defined in (43), and let $z_0 \in [z]$ be the complex representative of $[z]$. The zero z will be called zero of type k if $\mathrm{rank}(\mathbf{A}(\mathrm{z_0})) = 4 - \mathrm{k},\ 0 \le \mathrm{k} \le 4$. A zero of type 4 $(\mathrm{rank}(\mathbf{A}(\mathrm{z_0})) = 0)$ will be called the spherical zero. It has the property that all $z \in [z_0]$ are zeros. A zero of type 0 will be called isolated zero. In this case $z = -(\mathbf{A}(z_0))^{-1}\mathrm{col(B(z_0))}$ is the only zero in $[z_0]$. We will also call a real zero an isolated zero.

Since the one–sided quaternionic polynomials also belong to the class we are considering, zeros of types 0 and 4 will in fact occur. See [15]. From the study of the quadratic case in the next section, we shall see that zeros of type 2 will also exist. By some more tests with $n = 4$, we found that all ranks (zero to four) are indeed possible for $\mathbf{A}$. In the next section we show, that for $n = 2$ the cases $\mathrm{rank}(\mathbf{A}) = 1$ (type 3) and $\mathrm{rank}(\mathbf{A}) = 3$ (type 1) are impossible.

Definition 4. Let p be any quaternionic polynomial of degree $n \ge 2$. By $\#Z(p)$ we understand the number of equivalence classes in $\mathbb{H}$ which contain zeros of p. We call this number, essential number of zeros of p.

By this definition, $p(z) := z^2 + 1$ has one essential zero, since **i** and $-$**i** are located in the same equivalence class.

All polynomials with real coefficients and degree n as well as all quaternionic, one-sided polynomials of degree n have at most n essential zeros, see [33, 16].

Theorem 7. Let p be a quaternionic, two–sided polynomial of degree n. Then, $\#Z(p)$, the essential number of zeros of p, is, in general, not bounded by n.

Example 3. Let $p(z) := a_3 z^3 b_3 + a_2 z^2 b_2 + a_1 z b_1 + c_0$, where

$$\begin{array}{lll} a_3 := (1,1,0,0), & b_3 := (-1,-1,-1,0), & c_0 := (2,0,0,0). \\ a_2 := (-1,0,1,1), & b_2 := (0,-1,0,1), & \\ a_1 := (0,-1,1,1), & b_1 := (1,0,0,1), & \end{array}$$

The polynomial p is of degree three and the essential number of zeros of p is five.

Conjecture Let p be a quaternionic two–sided polynomial of degree n of the form (43). Then, the essential number of zeros of p will not exceed $2n$:

$$\#Z(p) \le 2n\,.$$

4.1 The quadratic case

In this section we will study the quadratic case

$$p(z) := z^2 + azb + c\,, \quad a,\ b,\ c \in \mathbb{H},\ a \notin \mathbb{R}, b \notin \mathbb{R}. \tag{51}$$

The cases $a \in \mathbb{R}$ or $b \in \mathbb{R}$ were already studied in [15]. We note, that it is not a restriction to assume that the highest coefficient (at z^2) is one. Let **I** be the 4×4 identity matrix. Then, for the quadratic case we have (use Definition (49) for $\mathbf{A}$ and (46) for $\mathbf{B}$ and $\alpha_0 = 0,\ \alpha_1 = 1,\ \alpha_2 = 2\Re(z),\ \beta_0 = 1, \beta_1 = 0,\ \beta_2 = -|z|^2$)

$$\mathbf{A}(z) = 2\Re(z)\mathbf{I} + \omega_3(a,b), \quad B(z) = c - |z|^2 \,. \tag{52}$$

We note here, that by Lemma 2 and by [13] the matrix $\mathbf{A}(z)$ is normal.

Lemma 6. The rank of the matrix $\mathbf{A}(z)$, defined in (52) can only be even, i. e. the rank can be zero, two or four.

Proof. Let $\mathrm{eig}(\mathbf{B})$ denote the column vector of all eigenvalues of a real square matrix $\mathbf{B}$. Then

$$\mathrm{eig}(\mathbf{A}(\mathrm{z})) = \mathrm{eig}(\omega_3(\mathrm{a},\mathrm{b})) + 2\Re(\mathrm{z}) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \,. \tag{53}$$

The matrix $\omega_3(a,b)$ is orthogonal (cf. Lemma 2) and all its eigenvalues have the same absolute value $|a||b|$. In particular, the eigenvalues of $\omega_3(a,b)$ are never zero. The four eigenvalues of $\omega_3(a,b)$ always come in two pairs, (a) either two pairs of complex conjugate numbers, or (b) one pair of complex conjugate numbers and one pair of the same real number or (c) two pairs of the same real number, where the two real pairs may only differ in sign. In case (a), $\mathbf{A}(z)$ will be non singular, in case (b) it may be non singular or have rank two. In case (c), $\mathbf{A}(z)$ may have rank zero, two or four. □

Theorem 8. For the zeros of a quadratic polynomial p defined in (51), there are the following possibilities:

1. All eigenvalues of $\omega_3(a,b)$ are nonreal. Then, only isolated zeros are possible.
2. There are real and complex eigenvalues. Then, isolated zeros or zeros of type 2 are possible.
3. All eigenvalues are real. Then, spherical zeros, zeros of type 2, and isolated zeros are possible.

Proof. Follows from the foregoing lemma. □

We will show, that spherical zeros are impossible if at least one of the coefficients $a,\ b$ in (51) is nonreal.

Lemma 7. Let $a,\ b \in \mathbb{H}$ and define $\mathbf{J} := \omega_3(a,b) := \omega_1(a)\omega_2(b)$, where $\omega_1,\ \omega_2$ are defined in (16), (17). Then, $\mathbf{J}$ has four identical real eigenvalues if and only if $a,\ b \in \mathbb{R}$.

Proof. Let $a,\ b \in \mathbb{R}$. Then $\mathbf{J} = ab\mathbf{I}$ and $\mathbf{J}$ has four identical real eigenvalues ab. Now, assume, that $\mathbf{J}$ has four identical real eigenvalues c. Put $a := (a_1,a_2,a_3,a_4),\ b := (b_1,b_2,b_3,b_4)$. In this case the characteristic polynomial is $\chi_{\mathbf{J}}(x) := \det(\mathbf{J} - x\mathbf{I}) = (x-c)^4$. It follows that $c^4 = \det(\omega_3(a,b)) = |a|^4|b|^4$, and thus, $c = \pm|a||b|$. It also follows that the trace is $\mathrm{tr}(\mathbf{J}) = 4a_1b_1 = 4c$. Therefore, $a_1b_1 = \pm|a||b|$. This implies $a, b \in \mathbb{R}$. □

Because of the eigenvalue formula (53), this lemma implies that the matrix $\mathbf{A}$ can have $\mathrm{rank}(\mathbf{A}) = 0$ only if $a,\ b \in \mathbb{R}$.

Example 4. Let $p(z) = z^2 + \mathbf{i}z\mathbf{j} + \mathbf{k}$. Some tests show that $z_0 = \frac{1}{2}(-1 + \mathbf{i}\sqrt{3})$ is a complex representative of a zero. In this case

$$\mathbf{A}(z_0) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad B(z_0) = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and $\operatorname{rank}(\mathbf{A}(z_0)) = 2$. The zeros $z = (z_1, z_2, z_3, z_4) \in [z_0]$ obey the equations

$$z_1 = \Re(z_0) = -0.5, \quad |z|^2 = |z_0|^2 = 1\,, \quad \mathbf{A}(z_0)\mathrm{col}(z) + B(z_0) = 0\,. \tag{54}$$

There are two solutions of type 2, namely

$$u_1 = \frac{1}{2}(-1,-1,1,1),\,, \quad u_2 = \frac{1}{2}(-1,1,-1,1)\,.$$

Example 5. Let $p(z) = z^2 + \mathbf{i}z\mathbf{j} + 1$. Then, there are two nonequivalent, complex representatives of zeros, namely $z_\pm = \frac{1}{2}(\pm 1 + \mathbf{i}\sqrt{3})$. For the representative $z_- = \frac{1}{2}(-1 + \mathbf{i}\sqrt{3})$ we have

$$\mathbf{A}(z_-) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad B(z_-) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $\operatorname{rank}(\mathbf{A}(z_-)) = 2$. The zeros $z = (z_1, z_2, z_3, z_4) \in [z_0]$ obey the equations (54). The solutions are

$$u_1 = \frac{1}{2}(-1,-1,1,-1),\,, \quad u_2 = \frac{1}{2}(-1,1,-1,-1)\,.$$

For the other representative $z_+ = \frac{1}{2}(\pm 1 + \mathbf{i}\sqrt{3})$ we have

$$\mathbf{A}(z_+) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B(z_+) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $\operatorname{rank}(\mathbf{A}(z_+)) = 2$. The zeros are

$$u_3 = \frac{1}{2}(1,-1,-1,-1),\,, \quad u_4 = \frac{1}{2}(1,1,1,-1)\,.$$

All together for the given quadratic polynomial there are four different equivalence classes which contain zeros of type 2 and two more isolated zeros $u_{5,6} = \frac{1}{2}(1 \pm \sqrt{5})\mathbf{k}$.

Example 6. Let $p(z) = z^2 + \mathbf{i}z\mathbf{j} + 1 + \mathbf{k}$. In this case we find that $z_0 := 1 + \mathbf{i}$ is a complex representative of a zero and we have

$$\mathbf{A}(z_0) = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 2 \end{pmatrix}, \quad B(z_0) = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and $\operatorname{rank}(\mathbf{A}(z_0)) = 4$. The isolated zero $z \in [z_0]$ is the unique solution of $\mathbf{A}(z_0)z + B(z_0) = 0$ which is

$$u = (1,0,0,-1)\,.$$

5 Quaternionic polynomials with multiple terms of the same degree

In the case of a simple polynomial with multiple terms of the same degree, we can sum the terms, for example

$$\tau_j(z) = a_j^{(1)} z^j + a_j^{(2)} z^j = (a_j^{(1)} + a_j^{(2)}) z^j =: a_j z^j\,,$$

and we obtain again the one–sided quaternionic polynomial (27).

In the two–sided case

$$t_j(z) = a_j^{(1)} z^j b_j^{(1)} + a_j^{(2)} z^j b_j^{(2)}$$

such simplification is not possible, nevertheless, the presented technique will also work.

Since one cannot combine terms of the same degree, we will specialize the general two–sided quaternionic polynomial defined in (39), (40) in the following way:

$$
\begin{aligned}
t_{jk}(z) &:= a_{0j}^{(k)}\cdot z\cdot a_{1j}^{(k)}\cdots a_{j-1,j}^{(k)}\cdot z\cdot a_{jj}^{(k)}\,, \quad k=1,2,\ldots,k_j,\ k_j\ge 0,\ k_n=k_0=1\,, && (55)\\
p(z) &:= \sum_{j=0}^{n}\sum_{k=1}^{k_j} t_{jk}(z)\,, \quad t_{n1}\neq 0,\ t_{01}\neq 0\,. && (56)
\end{aligned}
$$

The condition $k_n = 1$ together with $t_{n1} \neq 0$ ensures that there is exactly one term with degree n which is not vanishing. This allows to normalize the highest term to z^n. According to Eilenberg and Niven, [5], this condition guarantees the existence of at least one zero. However, the following development will also work if we have several terms of the highest degree, thus, allowing $k_n \geq 1$. The condition $k_0 = 1$ is not a restriction, since the constant terms could be combined to one term. The condition $t_{01} \neq 0$ implies that the origin $z = 0$ is never a zero.

We apply the column operator to p, again using the representation $z^j = \alpha_j z + \beta_j$, developed in (22) to (25) and the matrix ω_3 defined in (20). We obtain

$$
\begin{aligned}
\mathrm{col(p(z))} &= \sum_{j=0}^{n}\sum_{k=1}^{k_j}\mathrm{col(t_{jk}(z))}, && (57)\\
\mathrm{col}(t_{jk}(z)) &= \mathrm{col}\left(a_j^{(k)} z^j b_j^{(k)}\right) = \mathrm{col(a_j^{(k)}(\alpha_j z+\beta_j)b_j^{(k)})} && (58)\\
&= \alpha_j \mathrm{col}(a_j^{(k)} z b_j^{(k)}) + \beta_j \mathrm{col(a_j^{(k)} z b_j^{(k)})} && (59)\\
&= \mathbf{A}_{jk}\mathrm{col(z)} + \mathrm{col(B_{jk})}\,, \quad \text{where} && (60)\\
\mathbf{A}_{jk} &= \alpha_j\omega_3(a_j^{(k)}, b_j^{(k)})\,, \quad B_{jk} = \beta_j a_j^{(k)} b_j^{(k)}\,. && (61)
\end{aligned}
$$

If we put

$$
\mathbf{A} := \sum_{j=0}^{n}\sum_{k=1}^{k_j}\mathbf{A}_{jk}\,, \quad B := \sum_{j=0}^{n}\sum_{k=1}^{k_j} B_{jk}\,, \tag{62}
$$

we obtain exactly the representation (50). The classification of the zeros, given in Definition 3 will still be valid, as well as the further statements in Section 3. But Theorem 8 on quadratic polynomials will, in general, not be true. A quadratic polynomial will read

$$
p(z) := z^2 + \sum_{k=1}^{K} a^k z b^{(k)} + c, \tag{63}
$$

and the corresponding matrix representation is

$$
\begin{aligned}
\mathrm{col(p(z))} &= \big(2\Re(z)\mathbf{I} + \sum_{k=1}^{K}\omega_3(a^{(k)}, b^{(k)})\big)\,\mathrm{col(z)} + \mathrm{col(c-|z|^2)} && (64)\\
&= \mathbf{A}\mathrm{col(z)} + \mathrm{col(B)}\,. && (65)
\end{aligned}
$$

In this case, the matrix $\mathbf{A}$ contains a sum of ω_3 matrices and Lemma 6 will not be valid any more. We have to apply the general case.

Example 7. Let

$$
p(z) = z^2 + azb + czd + e\,, \quad a,b,c,d,e \in \mathbb{H}\,. \tag{66}
$$

Let us classify the zeros of this polynomial for three different choices of the coefficients.

Case 1.

$$a := (0,-1,0,0),\quad b := (0,1,1,0),\quad c := (0,1,-1,1),$$
$$d := (0,1,1,0),\quad e := (0,3,-1,3)\,.$$

In this case $z_0 = 1 + \mathbf{i}\sqrt{6}$ is a complex representative of a zero of p,

$$\mathbf{A}(z_0) = \begin{pmatrix} 3 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 3 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},\quad \mathrm{col}\,(B(z_0)) = \begin{pmatrix} -7 \\ 3 \\ -1 \\ -3 \end{pmatrix},$$

$\mathrm{rank}\,(\mathbf{A}(z_0)) = 2$ and there are two different zeros of type $k = 2$ of p in $[z_0]$, namely

$$u_1 := (1,-2,2,2)\,,\qquad u_2 := (1,-1,1,2)\,.$$

Case 2. Let us choose

$$a := (0,1,1,0),\quad b := (1,0,-1,0),\quad c := (0,0,1,1),$$
$$d := (0,1,1,0),\quad e := (16,4,-16,6)\,.$$

In this case $z_0 = 1 + \mathbf{i}\sqrt{29}$ belongs to a class of zeros of p,

$$\mathbf{A}(z_0) = \begin{pmatrix} 2 & -2 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & -2 \\ -2 & -2 & 0 & 2 \end{pmatrix},\quad \mathrm{col}\,(B(z_0)) = \begin{pmatrix} -14 \\ 4 \\ -16 \\ 6 \end{pmatrix},$$

$\mathrm{rank}\,(\mathbf{A}(z_0)) = 3$ and we have to solve $\mathbf{A}\mathrm{col(z)} + \mathrm{col(B)} = 0$ for $z \in [z_0]$. The only zero of type $k = 1$ of p in $[z_0]$ is

$$u := (1,-2,3,-4)\,.$$

Case 3.

$$a := (-4,-1,4,2),\quad b := (3,-3,3,-3),\quad c := (0,-5,0,-1),$$
$$d := (4,-3,-5,1),\quad e := (258,208,239,220)\,.$$

In this case $z_0 = 2 + \mathbf{i}\sqrt{83}$ is a complex representative of a zero of p,

$$\mathbf{A}(z_0) = \begin{pmatrix} -31 & -12 & -1 & 20 \\ -34 & -21 & 0 & 29 \\ -1 & -20 & -9 & 36 \\ 48 & -19 & -34 & 29 \end{pmatrix},\quad \mathrm{col}\,(B(z_0)) = \begin{pmatrix} 171 \\ 208 \\ 239 \\ 220 \end{pmatrix},$$

$\mathrm{rank}\,(\mathbf{A}(z_0)) = 4$ and there is one isolated zero (type $k = 0$) of p in $[z_0]$.

Let us note that if in (66) $b = c = 1$ then we put $z := u - d$ and obtain the one–sided case (see [29])

$$\tilde{p} := p(u-d) := u^2 + (a-d)u - ad + e\,,\quad z := u - d\,.$$

The investigation of this section makes also sense for the linear case, $n = 1$, if we would delete the condition $K := k_1 = 1$, i.e. the polynomial may have two or more linear terms. In this case we would have

$$p(z) \;:=\; \sum_{k=1}^{K} a^{(k)} z b^{(k)} + c\,, \tag{67}$$

$$\mathrm{col(p(z))} \;=\; \mathbf{A}\mathrm{col(z)} + \mathrm{col(c)},\quad \text{where}\quad \mathbf{A} = \sum_{\mathrm{k=1}}^{\mathrm{K}} \omega_3(\mathrm{a^{(k)}}, \mathrm{b^{(k)}})\,. \tag{68}$$

Since $\mathbf{A}$, $\mathrm{col(c)}$ do not depend on z, the equivalence classes have to be replaced with the full space $\mathbb{H}$. A zero of type k, $0 \le k \le 4$, is then a zero in a $k-$dimensional subspace of $\mathbb{H}$. Because of the loosening of the condition $k_1 = 1$, the equation $p(z) = 0$ may have no solution, like e.g. $p(z) := az - za + 1$. The linear case, also for systems is treated in more detail in Janovská and Opfer, 2008, [15].

5.1 Numerical computation of zeros

The representation of a given quaternionic, two-sided polynomial p in the form

$$P(z) := \mathbf{A}(z)z + B(z)$$

which was already used for the classification of the zeros can also be applied successfully to finding the zeros, by applying Newton's method to $P(z) = 0$. It shows the typical feature, that it may be slow in the beginning, but it will terminate then very quickly with quadratic rate.

One can show that both one– and two–sided quaternionic polynomials can have no multiple zeros (apart of some trivial examples). This is the reason why Newton's method works well.

In short, the application of Newton's method results in solving the following linear equation for s, repeatedly:

$$P(z) + P'(z)s = 0; \quad z := z + s, \tag{69}$$

where in the beginning one needs an initial guess z. In order to compute the (4×4) Jacobi matrix P' we use numerical differentiation. Let $e_k,\ k = 1, 2, 3, 4$ be one of the four standard unit vectors in $\mathbb{R}^4$, $z := (z_1, z_2, z_3, z_4)$. Then,

$$\frac{\partial P}{\partial z_k}(z) \approx \frac{P(z + he_k) - P(z)}{h}, \quad k = 1, 2, 3, 4, \quad h \approx 10^{-7}, \tag{70}$$

$$P'(z) := \left(\frac{\partial P}{\partial z_1}(z), \frac{\partial P}{\partial z_2}(z), \frac{\partial P}{\partial z_3}(z), \frac{\partial P}{\partial z_4}(z)\right). \tag{71}$$

The choice $h \approx 10^{-7}$ is the standard choice for computers with machine precision of $\approx 10^{-15}$. This choice implies a good balance between the round off and truncation errors.

If we apply the numerical techniques presented to the previous examples we obtain:
Example 4: There are two further non equivalent zeros both of type 4. The essential number of zeros is 3.
Example 5: There are two further non equivalent zeros both of type 4. The essential number of zeros is 4.
Example 6: There is one more zero of type 4, and there are two more equivalent zeros of type 2. The essential number of zeros is 3.
An application to Example 7, reveals in (a) two additional, non equivalent zeros, in (b) one additional non equivalent zero, in (c) one additional non equivalent zero. Thus, the essential number of zeros is in (a): three zeros, in (b) two zeros, in (c) two zeros.
Therefore, in several examples, the essential number of zeros exceeds the degree.

6 Sylvester's equation in quaternions

Let us investigate Sylvester's equation in quaternions. In this case $n = 1$ and the equation has the form

$$az + zb = e\,, \quad z,\, a,\, b,\, e \in \mathbb{H}\,. \tag{72}$$

Our aim is to find zeros of the linear two–sided quaternionic polynomial, we will call it Sylvester's polynomial,

$$p(z) := az + bz - e\,, \quad z,\, a,\, b,\, e \in \mathbb{H}\,. \tag{73}$$

In this case, $p(z)$ is linear, i.e. $n = 1$, two–sided quaternionic polynomial with two linear terms, i.e. we follow the theory of the two–sided quaternionic polynomials (67), (68). Because $K := k_1 := 2$, i.e. there are two terms of the highest degree $n = 1$, the existence of the solution is not guaranteed. We have $p(z) = a^{(1)}zb^{(1)} + a^{(2)}zb^{(2)} + c$ where

$$a^{(1)} := a, \quad b^{(1)} := (1, 0, 0, 0), \quad a^{(2)} := (1, 0, 0, 0), \quad b^{(2)} := b, \quad c := -e\,.$$

Then

$$
\begin{aligned}
P(z) &:= \mathrm{col(p(z))} = \mathbf{A}\mathrm{col(z)} + \mathrm{col(c)}, \quad \text{where} \\
\mathbf{A} &= \omega_3(a^{(1)}, b^{(1)}) + \omega_3(a^{(2)}, b^{(2)}) = \omega_1(a) + \omega_2(b)\,.
\end{aligned}
$$

Example 8. Let Sylvester's equation $az + zb = e$ be given, where

$$a := (1, -1, 1, 1)\,, \quad b := (1, 1, 1, 1)\,, \quad e := (-4, 4, 8, 0)\,.$$

Then

$$\mathbf{A} = \omega_1(a) + \omega_2(b) = \begin{pmatrix} 2 & 0 & -2 & -2 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 0 & -2 & 2 \end{pmatrix}.$$

The matrix **A** doesn't depend on z, $\mathrm{rank}\,(\mathbf{A}(z_0)) = 4$ and there is one isolated zero (type $k = 0$) of p. This can be found as a solution of the system of linear equations

$$P(z) := \mathrm{col(p(z))} = \mathbf{A}\mathrm{col(z)} - \mathrm{col(e)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is $z = (1, 2, 2, 1)$.

The linear quaternionic polynomial as well as Sylvester's equation are treated in detail in [22]. Let us repeat here only the following theorem.

Theorem 9. *Let $a := (a_1, a_2, a_3, a_4)$, $d := (d_1, d_2, d_3, d_4)$. The Sylvester equation $ax + xd = e$ has a unique solution for all choices of $e \in \mathbb{H}$ if and only if $\sum_{j=2}^{4}(a_j^2 - d_j^2) \neq 0$ or $a_1 + d_1 \neq 0$.*

If the unique solution exist it is given by

$$x = f_l^{-1}(e + a^{-1}e\overline{d}), \quad f_l := 2\Re d + a + |d|^2 a^{-1} \quad \text{if } a \neq 0, \text{ or} \tag{74}$$

$$x = (e + \overline{a}ed^{-1})f_r^{-1}, \quad f_r := 2\Re a + d + |a|^2 d^{-1} \quad \text{if } d \neq 0. \tag{75}$$

Corollary Let a, d be arbitrary quaternions. Then, Sylvester's polynomial $p(x) = ax + xd$ is singular (has no solution or many solutions) if and only if

$$|a| = |d| \text{ and } \Re a + \Re d = 0, \tag{76}$$

or in other words if and only if a and $-d$ are equivalent.

Example 9. Let us try to find zero of the Sylvester's polynomial $p(x) = ax - xa - e$ where $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$, $e := (-1, 0, 0, 0)$. Following the previous corollary we know that there is no unique solution of the corresponding Sylvester's equation. If we use the theory of the two–sided quaternionic polynomials, we found out that

$$\mathbf{A} = \omega_1(a) + \omega_2(-a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2a_4 & 2a_3 \\ 0 & 2a_4 & 0 & -2a_2 \\ 0 & -2a_3 & 2a_2 & 0 \end{pmatrix}.$$

Then the $\mathrm{rank}(\mathbf{A}) = 2$ and the equation

$$P(z) := \mathrm{col(p(z))} = \mathbf{A}\mathrm{col(z)} + \mathrm{col(e)}$$

has no solution, because for the resulting system of linear equations $P(z) = 0$ the Frobelius theorem is not valid.

7 Conclusions

We investigated polynomials with quaternionic coefficients both of the simple and two–sided type.

The quaternionic polynomials of the simple type may have two different types of zeros: isolated zeros and spherical zeros. We gave a characterization of the zeros and, based on this characterization, we presented the algorithm for producing all zeros including their types. For more details see [15].

The two–sided quaternionic polynomials may have all together five different types of zeros. These types are defined by the rank of a certain real 4×4 matrix. This information can also be used to find all zeros in the same similarity class (13) if only one zero in that class is known. The essential tool is the description of the polynomial p by a matrix equation $P(z) := \mathbf{A}(z)z + B(z)$, where $\mathbf{A}(z)$ is a real 4×4 matrix determined by the coefficients of the given polynomial p and $P,\ z,\ B$ are real column vectors with four rows. This representation allows also to include two-sided quaternionic polynomials which contain several terms of the same degree and to prove, that the essential number of zeros of a quaternionic, two-sided polynomial p of degree n is, in general, not bounded by n. Our conjecture is that the bound is $2n$. More details can be found in [16].

It turned out that Newton's method applied to the equation $P(z) = 0$ is a very effective tool in finding the zeros. For Newton's method, see also [23].

Finally, we apply the theory to the linear two–sided quaternionic Sylvester's polynomial. Sylvester's equation was also treated by a little different technique in [19, 25]. There are other algebraic equations in quaternions that can be treated by the technique presented here, e.g. algebraic Riccati equation, or algebraic Bernoulli equation.

We would like to mention, that we used two essential ideas of other authors, namely
– The idea by Anatoliy Pogoruĭ and Michael Shapiro, [33], was to write the powers z^j in the form $\alpha\, z + \beta$ with real α, β, which reduces the two-sided polynomials to a sum of terms of the form azb. This idea also gave birth to the introduction of equivalence classes of zeros in $\mathbb{H}$.
– Another idea, by Ludmilla Aramanovitch, [1], was the introduction of the matrix ω_2 (formula (17)) which permitted to pull out the variable z from azb. Both ingredients allowed the development of the important formula (48).

Because there is an isomorphism between the skew field of quaternions $\mathbb{H}$ and certain matrices in $\mathbb{C}^{2\times 2}$, let us denote it for a while by $\mathbb{H}_{\mathbb{C}}$, and also between $\mathbb{H}$ and certain matrices in $\mathbb{R}^{4\times 4}$, denoted by $\mathbb{H}_{\mathbb{R}}$. both simple and two–sided quaternionic polynomials can be understood as matrix polynomials with all coefficients either from $\mathbb{H}_{\mathbb{C}}$ or from $\mathbb{H}_{\mathbb{R}}$, [16]. However, there is one difference. If we solve a particular matrix equation, we obtain zeros in the considered matrix space. Experiments showed, that these matrix polynomials have the wanted quaternionic zeros but may in addition have other zeros which lack an interpretation as a quaternion.

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1983 – 1998 Scientific co-worker at Department of Numerical Analysis, Faculty of Mathematics and Physics, Charles University, Prague
1998 – 2004 Assistant Professor at Department of Mathematics, Faculty of Chemical Engineering, Institute of Chemical Technology, Prague
2004 – till now Associate Professor at Department of Mathematics, Faculty of Chemical Engineering, Institute of Chemical Technology, Prague

In the course of my academical career I have lectured on various topics - ranging from pure mathematics to the programming languages. I would like to recall e.g. Partial Differential Equations and Numerical Linear Algebra for MSc. students at Faculty of Maths&Physics, Charles University, Prague, Mathematics for chemical engineers and Methods of Applied Mathematics (mathematical foundation of the finite element method) for MSc. and Ph.D. students at Institute of Chemical Technology, Prague.

My research field is Numerical Analysis, namely numerical solution of nonlinear problems, bifurcations and problems in Numerical Linear Algebra.

For many years I have cooperated with my colleagues abroad, namely in Germany (University Hamburg, University Marburg, University Konstanz) and in Japan (Institute of Statistical Mathematics, Waseda University, Tokyo). I am doing my best to take advantage of these contacts for the benefit of our students in the frame of the Socrates-Erasmus program.

I have been a member of several local organizing committees for international conferences in Prague (annual GAMM meeting - 1996, Czech-American Workshop on Iterative Methods and Parallel Computing - 1997, 3rd Scientific Colloquium - 2001, ICT Prague, Software and Algorithms of Numerical Mathematics - every second year till 2004). Currently, I am a member of the scientific committee of the international conference Preconditioning of Iterative Methods, which will be held in Prague in July 2013.

Taking into account that an academic job should also be considered as a hobby these days, there is no much time left for my really private pleasures. But there are some: literature, classical music and theater.

Future scientific research

1. **Numerical Linear Algebra**
   As I have already mentioned in Conclusions, there is an isomorphism between quaternions and special 2×2 complex matrices and also an isomorphism between quaternions and special 4×4 real matrices, i.e. the quaternionic polynomials can be understood as certain matrix polynomials. Naturally, our nearest aim is classification and computation of zeros of matrix polynomials. It is truly a difficult task. So far we are able fully classify only zeros of quadratic matrix polynomials with general matrices 2×2. I have presented some intermediate results for the matrix polynomials on the 25th Biennial Conference on Numerical Analysis in Glasgow 2011. Together with G. Opfer, we have submitted the paper "On a Toeplitz form of the characteristic polynomials with an application to matrix polynomials" to Numerische Mathematik.

2. **Nonlinear Dynamical Systems**
   We study the long–term qualitative behavior of nonlinear dynamical systems including numerical solutions and simulations. In particular, we apply the Filippov systems theory to selected problems from biology and chemical engineering. For example we explored a new formulation of Bazykin's ecological model, a predator-prey model with human intervention, an ideal closed gas–liquid system including DAE formulation of the system with a chemical reaction, etc. We illustrate the theory by simulations of the behavior of the specific systems.

Concept of teaching

Except for the basic courses for bachelors, I run advance lectures that are designed for master and doctoral students who want to deepen their mathematical education: the objective is to model, simulate and solve particular problems in chemical engineering. This includes Dynamical Systems, Ordinary and Partial Differential equations, Finite Element Method, Numerical Analysis, etc. These activities are directly related to my research.

Selected publications

Impacted international journals

1. Janovská D., Marek I.: Once more about the monotonicity of the Temple quotient, Applications of Mathematics 29 (1984), 321 – 339.

2. Janovská D.: Numerical treatment of subspace-breaking Takens-Bogdanov points with nonlinear degeneracies, SIAM J.Numer.Anal. 31, No. 5 (1994), 1415 – 1433.

3. Janovská D., Böhmer K., Janovský V.: Numerical analysis of the imperfect bifurcation, ZAMM (1997), 445 - 448.

4. Böhmer K., Janovská D., Janovský V.: Computer aided analysis of imperfect bifurcation diagrams, East-West J. Numer. Math 6, No. 3 (1998), 207 – 222.

5. Korbelář J., Janovská D: Numerical model of a pine in a wind, Applications of Mathematics 44, No. 6 (1999), 459 – 468.

6. Janovská D., Opfer G.: A Note on Hyperbolic Transformations, NLAA 8 (2001), 127 – 146.

7. Janovská D.: Decomposition of an updated correlation matrix via hyperbolic transformations, Applications of Mathematics 47, No. 2 (2002), 101 – 113.

8. Böhmer K., Janovská D., Janovský V.: On the numerical analysis of the imperfect bifurcation of $\mathrm{codim} \leq 3$, SIAM J. Numer. Anal 40, No. 2 (2002), 416 – 430.

9. Böhmer K., Janovská D., Janovský V.: Computing differentential of an unfolded contact diffeomorphism, Applications of Mathematics 48, No. 1 (2003), 3 – 30.

10. Janovská D., Opfer G.: Givens' transformation applied to quaternion valued vectors, BIT Numerical Mathematics 43, No.5 (2003), 991 – 1002.

11. Janovská D., Opfer G.: Givens transformation for quaternion-valued matrices applied to Hessenberg reduction, ETNA, Electronic Transactions on Numerical Analysis, Vol.20, 2005, pp.1 – 26.

12. Janovská D., Opfer G.: Computing quaternionic roots by Newton's method, Electronic Transactions on Numerical Analysis (ETNA), Vol.26 (2007), pp.82 – 102.

13. Janovská D., Opfer G.: Linear Quaternionic Systems, Mitt. Math. Ges. Hamburg 27 (2008), 223 – 234.

14. Janovská D., Opfer G.: Decompositions of quaternions and their matrix equivalents. In: Matrix Methods: Theory, Algorithms, Applications, V. Olshevsky, and E. Tyrtyshnikov eds., World Scientific Publishing Company (2010), 20 – 30.

15. Janovská D., Opfer G.: A note on the computation of all zeros of simple quaternionic polynomials, SIAM J. Numer. Anal. 48 (2010), 244 – 256.

16. Janovská D., Opfer G.: The classification and the computation of the zeros of quaternionic, two-sided polynomials, Numerische Mathematik, Volume 115, No.1 (2010), 81 – 100.

17. Janovská D., Janovský V.: The analytic SVD: On the non-generic points on the path, ETNA (Electronic Transaction on Numerical Analysis), Vol. 37 (2010), 70 – 86.

18. Janovská D., Opfer G.: The Nonexistence of Pseudoquaternions in $\mathbb{C}^{2\times 2}$, Advances in Applied Clifford Algebras 21 (2011), 531 – 540. DOI 10.1007/s00006-010-0273-1.

19. Janovská D., Janovský V., Tanabe K.: A note on computation of pseudospectra, Mathematics and Computers in Simulation, Elsevier, DOI 10.1016/j.matcom.2012.03.002.

Proceedings of international conferences

1. Böhmer K., Janovská D., Janovský V.: A postprocessing analysis for wing cusp singularity, Proceedings of SANM'99, Nečtiny, 1999, 29 – 36.

2. Janovská D., Janovský V.: A postprocessing procedure for symmetry-breaking bifurcation (a case study), Proceedings of the 3rd Colloquium of ICT, Prague, 2001, 27 – 37.

3. Janovská D., Janovský V.: A postprocessing of Hopf bifurcation points, Proceedings of ENUMATH, Prague, 2003, Springer Verlag Berlin Heidelberg, 2004, 502 – 509.

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7. Hanus T., Janovská D.: Discontinuous, piecewise-smooth dynamical systems, CHISA 2006, Praha, Czech Republic, CD-ROM of Full Texts CHISA 2006, ISBN 80-86059-45-6, Title Index: D, Org.No.: F4.8.

8. Janovská D., Opfer G.: Hamilton meets Newton, on quaternionic roots. Proceedings of 17th International Conference on the Applications of Computer Science and Mathematics in Architecture and Civil Engineering, K. Gürlebeck, C. Könke, eds., Weimar (2006), 57 – 61.

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