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Oscilatorické vlastnosti řešení diferenciálních rovnic

Oscillatory properties of solutions of differential equations

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Summary

A unified approach to the study of oscillatory properties of solutions to ordinary, partial and abstract differential equations of evolution is presented. The definition of oscillations involves both qualitative and quantitative aspects of this phenomenon. One of the quantitative characteristic of oscillations is the so-called oscillatory time. The concept of oscillatory time is defined for real functions as well as for abstract functions with values in a Banach space. The conditions for the second order evolutionary differential equations are derived to ensure the oscillatory character of solutions. At the same time, estimates of the oscillatory time are provided by means of the so-called summit function. A great deal of emphasis is given to the presence of damping terms in equations resulting from various dissipation mechanisms in systems. The theory is applied to the generalized Liénard equations, to the wave equation with two kinds of dissipation and to the Euler–Bernoulli beam equation with dissipative terms.

Souhrn

Je ukázán jednotný přístup ke studiu oscilatorických vlastností řešení obyčejných, parciálních a abstraktních evolučních diferenciálních rovnic. Definice oscilací zahrnuje jak kvalitativní tak i kvantitativní aspekty tohoto jevu. Jednou z kvantitativních charakteristik oscilací je tzv. oscilatorický čas. Pojem oscilatorického času je definován pro reálné funkce i pro abstraktní funkce s hodnotami v Banachově prostoru. Pro evoluční diferenciální rovnice druhého řádu jsou odvozeny podmínky zajišťující oscilatorický charakter řešení. Zároveň jsou poskytnuty odhady oscilatorického času pomocí tzv. vrcholové funkce. V rovnicích je důraz kladen na přítomnost tlumících členů, které vznikají v důsledku různých disipativních mechanismů v systémech. Teorie je aplikována na zobecněné Liénardovy rovnice, na vlnovou rovnici s dvěma druhy disipace a Eulerovu–Bernoulliovu rovnici tyče s disipativními členy.

Klíčová slova

Oscilatorická řešení obyčejných, parciálních, abstraktních evolučních diferenciálních rovnic, oscilatorický čas, disipativní mechanismy, zobecněné Liénardovy rovnice, disipativní vlnová rovnice, Euler–Bernoulliova rovnice tyče

Keywords

Oscillatory solutions of ordinary, partial, abstract evolution differential equations, oscillatory time, dissipative mechanisms, generalized Liénard equations, dissipative wave equation, Euler–Bernoulli beam equation

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1. Oscillatory functions

Oscillatory phenomena are encountered in various fields of science, engineering and actually in everyday life. The theoretical study of oscillations dates back to Pythagoras in the 6th century B.C. We use an appropriate definition of oscillation that includes both qualitative and quantitative aspects of this phenomenon. A quantitative characteristic of oscillations is the so-called oscillatory time.

1.1. Oscillatory time

Let the evolution of a quantity we are interested in is described by a real continuous function $t \mapsto u(t)$ where the time variable t runs through the interval $J_0 = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$ or $J_0 = \mathbb{R}$. The function u is called *oscillatory (about zero at* $+\infty$*)*, alternately said, u *oscillates (about zero if* t *tends to infinity)*, if there exists a positive constant θ, called *oscillatory time,* such that u assumes both positive and negative values in any interval $J \subset J_0$ the length of which is greater than θ, $|J| > \theta$ in short. In the sequel, we say briefly that u is oscillatory.

An alternative definition is possible: the function u is oscillatory if and only if there exists $\theta > 0$ such that for any interval $J \subset J_0$, $|J| > \theta$, there are t_1, $t_2 \in J$ with $u(t_1) < 0 < u(t_2)$ and, consequently, $u(t_0) = 0$ for some $t_0 \in J$. The assertion concerning the root t_0 of the function u follows fundamentally from the continuity of the function u. However, the concept of oscillatory function can be generalized, by means of the former definition, to a function that is merely measurable and not necessarily continuous, as we can see in Sec. 3.1.

Classical examples of oscillatory functions are continuous periodic functions with the mean value zero. By the mean value (average) of a periodic function u with the period T we mean the number $T^{-1}\int_0^T u(t)\,dt$. With the usual appropriate definition of the mean value the same is true also for real almost periodic functions (see [7]).

Oscillatory functions arise from mathematical models in physics, chemistry, biology, economics, sociology, *etc.*, defined by differential equations of evolution. The first two following examples concern the linear oscillation theory. The oscillatory functions appear here as solutions of linear ordinary differential equations of second order and the explicit form of these functions is well-known. Moreover, we can readily see the oscillatory time of these solutions. It is important that there are methods how to establish the estimate of the oscillatory time even for solutions of equations, such as (1.2.1), whose explicit formulæ are not available. The oscillatory time that can be chosen the same for all non-constant solutions of the given equation is called *uniform oscillatory time* and the equation is called *uniformly oscillatory* in such instances.

1.2. Examples

We start with some examples concerning ordinary differential equations governing systems with single degree of freedom. For examples with partial differential equations (distributed parameter systems) see Sec. 4.

EXAMPLE 1.2.1.
Harmonic oscillator. If the motion of the system is harmonic (e. g. electric LC circuit without resistance, in mechanics undamped spring-mass system) the corresponding model is represented by the equation $\ddot{u}(t) + q\,u(t) = 0$ where $q > 0$. All regimes of this system (harmonic oscillator) are stationary: the zero is the equilibrium state and all other solutions are given by the formula $u = A\sin(\sqrt{q}\,t + \varphi)$ where $A > 0$ and φ are arbitrary real constants. These functions are periodic with the same period $2\pi/\sqrt{q}$ and the zero mean value. The equation is uniformly oscillatory since the number $\theta = \pi/\sqrt{q}$ is the common oscillatory time of all non-constant (non-zero) solutions.

EXAMPLE 1.2.2.
Linear oscillations with damping or amplification. If in the above system a damping (or amplification) is present the corresponding equation assumes the form $\ddot{u}(t) + 2p\,\dot{u}(t) + q\,u(t) = 0$ where $p > 0$ (or $p < 0$). Under the assumption $p^2 < q$ the equation is uniformly oscillatory since the formula for the general solution $u = Ae^{-pt}\sin(\sqrt{q - p^2}\,t + \varphi)$, where $A \geq 0$ and $\varphi \in \mathbb{R}$ are arbitrary, yields that all non-zero solutions are oscillatory with the common oscillatory time $\theta = \pi/\sqrt{q - p^2}$.

EXAMPLE 1.2.3.
Nonlinear oscillations. Let us consider the nonlinear equation

$$\ddot{u} + f(t, \dot{u}) + g(t, u) = 0, \quad t \in J_0, \tag{1.2.1}$$

with given functions f and g, $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, $t \in J_0$ (and satisfying sufficient smoothness and growth conditions in order that the existence and uniqueness of global (defined on J_0) solutions of the corresponding initial value problem be ensured). Eq. (1.2.1) is uniformly oscillatory provided that there exist constants q and M such that:

- $u\,g(t, u) \geq q\,u^2, \qquad t \in J_0,\ u \in \mathbb{R},$
- $|f(t, v)| \leq 2M\,|v|, \qquad t \in J_0,\ v \in \mathbb{R},$
- $q > 0,\ 0 \leq M < \sqrt{q}.$

The evaluation of the uniform oscillatory time (in some sense optimal) can be established in terms of quantities q and M by means of the so-called summit function (see Remark 2.1.1). A better estimate is possible if, moreover, the "sign" assumption (the sign of $f(t, v)$ is the same as that of v) is fulfilled:

- $f(t, v)\,v \geq 0, \qquad t \in J_0,\ v \in \mathbb{R}.$

2. Auxiliary functions and sets

We introduce two functions (summit function, universal comparison function) that prove useful in the study of oscillatory properties, in particular, for estimating the dissipative terms in differential equations. Moreover, the summit function is used for the evaluation of the quantitative measure of oscillations, that is, the oscillatory time (for more details see [14]). To facilitate the formulation of oscillation assumptions we adopt, in Sec. 2.3, a suitable notation for a union of two cones in the plane with a common vertex at the origin (see [15]).

2.1. The summit function

Let us denote

$$\mathcal{O} = \left\{\, (q,p) \in \mathbb{R}^2; q > 0,\, p > -\sqrt{q}\,\right\}. \tag{2.1.1}$$

The set $\mathcal{O}$ is the set of couples $(q,p) \in \mathbb{R}^2$ for which solutions of the equation $\ddot{u} + 2p\,\dot{u}^+ + q\,u = 0$, $t \in \mathbb{R}$, admit positive local maxima. Here $\dot{u}^+(t) = \max\{\dot{u}(t),\, 0\}$. The summit function is a function $(q,\, p) \mapsto \vartheta_p^q$, $\vartheta_p^q\colon \mathcal{O} \to \mathbb{R}$, that to any $(q,\, p)$ assigns the first positive t where the maximum of the solution satisfying $u(0) = 0$, $\dot{u}(0) = u_1$ (> 0) is attained (the value of ϑ is independent of u_1). The summit function is continuous on $\mathcal{O}$ and monotonically decreasing in each variable while the other is fixed. Its explicit form is:

$$\vartheta_p^q = \begin{cases} \dfrac{\pi}{\sqrt{q-p^2}} + \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}\,, & -\sqrt{q} < p < 0, \\ \dfrac{\pi}{2\sqrt{q}}\,, & p = 0, \\ \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}\,, & 0 < p < \sqrt{q}, \\ \dfrac{1}{\sqrt{q}}\,, & p = \sqrt{q}, \\ \dfrac{1}{\sqrt{p^2-q}} \operatorname{argtanh} \dfrac{\sqrt{p^2-q}}{p}\,, & p > \sqrt{q}. \end{cases} \tag{2.1.2}$$

Remark 2.1.1.
The oscillatory time in Example 1.2.3 can be shown to be $\theta = 2\vartheta_{-M}^q$. Under the validity of the sign assumption it holds $\theta = \vartheta_{-M}^q + \vartheta_0^q$ (see [12], [29]). Let $g(t,u) = qu$ with $q > 0$. Now, if $f(t,v) = 2pv$ for some $p \in \mathbb{R}$, $|p| = M < \sqrt{q}$ (the linearly damped (or amplified) oscillator) then $\theta = \vartheta_{-M}^q + \vartheta_M^q$, that is $\theta = \pi/\sqrt{q-M^2}$ and this is just the formula stated in Example 1.2.2. In particular, for $M = 0$ we get again the result for the harmonic oscillator from Example 1.2.1, namely $\theta = 2\vartheta_0^q = \pi/\sqrt{q}$. Note that $\pi/\sqrt{q-p^2} < \vartheta_{-M}^q + \vartheta_0^q < 2\vartheta_{-M}^q$.

2.2. The universal comparison function

First, let us define an auxiliary function (stemmed from the linearly damped oscillation theory) $(t,q,p) \mapsto E(t,q,p)$, $E(t,q,p)\colon \mathbb{R}\times\mathcal{O}\to\mathbb{R}$, by the formula

$$E(t,q,p) = \begin{cases} \dfrac{1}{\sqrt{q-p^2}}\,\exp(-p\,t)\,\sin(\sqrt{q-p^2}\,t)\,, & -\sqrt{q}<p<\sqrt{q},\\ t\exp(-\sqrt{q}\,t)\,, & p=\sqrt{q},\\ \dfrac{1}{\sqrt{p^2-q}}\,\exp(-p\,t)\,\sinh(\sqrt{p^2-q}\,t)\,, & p>\sqrt{q}. \end{cases}$$

The universal comparison function C is a real function $(t,q,p,n)\mapsto C(t,q,p,n)$ defined for $t\in\mathbb{R}$, $(q,p)\in\mathcal{O}$ and $(q,n)\in\mathcal{O}$ as follows:

$$C(t,q,p,n) = \begin{cases} E(t,q,p), & t\in[0,\vartheta^q_p],\\ \exp(-p\,\vartheta^q_p + n\,\vartheta^q_n)\ E(\vartheta^q_p+\vartheta^q_n-t,q,n), & t\in(\vartheta^q_p,\ \vartheta^q_p+\vartheta^q_n]. \end{cases} \tag{2.2.1}$$

The function $t\mapsto c(t)=C(t,q,p,n)$ with q, p, n fixed, has the following properties:

- $c\in C^2([0,\ \vartheta^q_p+\vartheta^q_n])$,
- $\ddot{c}+2(p\,\dot{c}^+ + n\,\dot{c}^-)+q\,c=0, \qquad t\in[0,\ \vartheta^q_p+\vartheta^q_n]$,
- $c(0)=c(\vartheta^q_p+\vartheta^q_n)=0, \qquad c(t)>0, \quad t\in(0,\ \vartheta^q_p+\vartheta^q_n)$,
- $\dot{c}(0)=1, \quad \dot{c}(t)>0, \quad t\in[0,\ \vartheta^q_p), \qquad \dot{c}(\vartheta^q_p)=0$,
- $\dot{c}(t)<0, \quad t\in(\vartheta^q_p,\ \vartheta^q_p+\vartheta^q_n], \quad \dot{c}(\vartheta^q_p+\vartheta^q_n)=-\exp(-p\,\vartheta^q_p+n\,\vartheta^q_n)$.

2.3. Cone sets in the plane

Let now p, n, P, N be arbitrary constants in $\mathbb{R}$, $p\le P$, $n\ge N$. The symbol ${}^N_n\mathbb{X}^P_p$ stands for the set (see Figure 1)

$${}^N_n\mathbb{X}^P_p = \{\,(u,y)\in\mathbb{R}^2;\ pu^+ - nu^- \le y \le Pu^+ - Nu^-\,\} \tag{2.3.1}$$

(since $u^\pm=\max\{\pm u,\,0\}$, $u=u^+-u^-$ and $|u|=u^++u^-$). In particular, for any $P>0$,

$${}^{-P}_{P}\mathbb{X}^P_{-P} = \{\,(u,y)\in\mathbb{R}^2;\ |y|\le P|u|\,\},\ \ {}^{0}_{P}\mathbb{X}^P_{0} = \{\,(u,y)\in\mathbb{R}^2;\ |y|\le P|u|,\ uy\ge 0\,\}. \tag{2.3.2}$$

Further, for p and N fixed we denote (see Figure 2)

$${}^N_\infty\mathbb{X}^\infty_p = \bigcup_{\substack{p\le P<\infty,\\ N\le n<\infty}} {}^N_n\mathbb{X}^P_p\,. \tag{2.3.3}$$

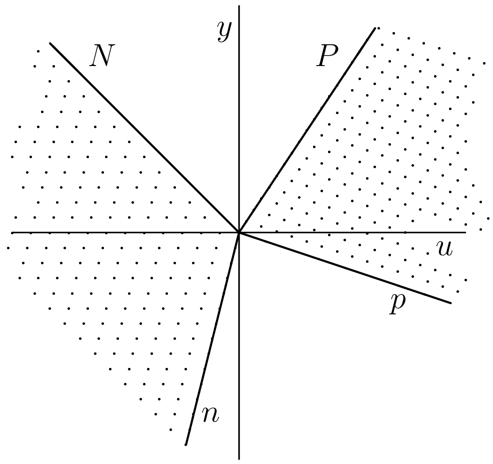

Figure 1: The set ${}_{n}^{N}\mathbb{X}_{p}^{P}$

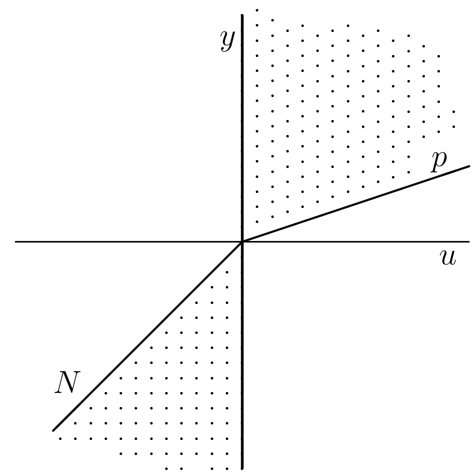

Figure 2: The set ${}_{\infty}^{N}\mathbb{X}_{p}^{\infty}$

3. Ordinary differential equations

We start with the definition of a measurable oscillatory function and the oscillatory time. In addition to the second order nonlinear ordinary differential equation mentioned in Sec. 1.2 we show (and prove briefly) the oscillation results concerning another second order nonlinear differential equation (generalized Liénard equations). For more details see [16].

3.1. Oscillatory function

A (Lebesgue) measurable function

$$t \mapsto u(t), \quad u\colon J_0 \to \mathbb{R}, \tag{3.1.1}$$

is called *oscillatory (about zero at* $+\infty$*)* if there exists (the so-called *oscillatory time*) $\theta > 0$ such that for any interval $J \subset J_0$, $|J| > \theta$, the function u changes the sign on J, more precisely, we have simultaneously

$$\text{meas}\{\, t \in J;\, u(t) > 0\} > 0 \quad \text{and} \quad \text{meas}\{\, t \in J;\, u(t) < 0\} > 0. \tag{3.1.2}$$

Provided u possesses the (pseudo-analyticity) property:

$$u = 0 \ \textit{a.e.in}\, J \subset J_0,\ J \textit{ compact (non-degenerate) interval} \implies u = 0 \ \textit{a.e.in}\, J_0, \tag{3.1.3}$$

an alternate definition is available. A function $u\colon J_0 \to \mathbb{R}$, $u \neq 0$ almost everywhere in J_0, is oscillatory if and only if there exists $\theta > 0$ such that for any interval $J \subset J_0$ the implication holds true:

$$u(t) \geq 0 \ \ (u(t) \leq 0,\ \textit{respectively}) \ \ \textit{a. e. in } J \Longrightarrow |J| \leq \theta. \tag{3.1.4}$$

For u being a solution of a differential equation the property (3.1.3) is fulfilled whenever the uniqueness of solutions of the initial value problem takes place.

3.2. Generalized Liénard equations

Let $J_0 = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$. We study the equation

$$\ddot{u} + f(t,u)\,\dot{u} + g(t,u) = 0, \qquad t \in J_0, \tag{3.2.1}$$

where $(t,u) \mapsto f(t,u)$, $(t,u) \mapsto g(t,u)$, $f\colon J_0 \times \mathbb{R} \to \mathbb{R}$, $g\colon J_0 \times \mathbb{R} \to \mathbb{R}$, $g(t,0) \equiv 0$, and assume that for any $u_0, u_1 \in \mathbb{R}$ the initial value problem given by Eq. (3.2.1) and by initial conditions $u(t_0) = u_0$ and $\dot{u}(t_0) = u_1$ has a unique maximal C^2-solution (or merely W^1_∞ (or $W^1_{r,loc}$ for some $r \geq 1$)-solution $t \mapsto u(t)$, $u\colon [t_0, t_{\max}) \to \mathbb{R}$, and no solution can vanish on a non-degenerate compact interval $J \subset [t_0, t_{\max})$ unless $u \equiv 0$ on J_0. Denote by $\mathcal{U}$ the set of solutions u for which $t_{\max} = +\infty$ (the so-called global solutions) and assume that $\mathcal{U} \neq \emptyset$.

We establish conditions on functions f and g under which there exists a number θ such that any function $u \in \mathcal{U}$ is oscillatory with the same oscillatory time θ. An estimate for the oscillatory time is given in terms of the summit function.

3.3. Oscillation assumptions

Denote

$$F(t,u) = \int_0^u f(t,\bar{u})\,d\bar{u}. \tag{3.3.1}$$

and $\tilde{g} = g - F_t$. Let us assume that there exist constants $q_\pm$, $m_\pm$ and $M_\pm$ such that

$$(q_\pm,\, -M_\pm) \in \mathcal{O}, \quad (q_\pm,\, m_\pm) \in \mathcal{O}, \tag{3.3.2}$$

where $\mathcal{O}$ is given in (2.1.1). The next assumptions concern the inclusion (uniform with respect to t) of graphs of functions $u \mapsto \tilde{g}(t,u)$ and $u \mapsto F(t,u)$ in the sets of type $\mathbb{X}$ (union of two cones with a common vertex at the origin) introduced in Sec. 2.3, namely,

$$\bigcup_{\substack{t\in J_0,\\ u\in\mathbb{R}}} \big(u,\, \tilde{g}(t,u)\big) \subseteq {}^{q_-}_{\infty}\mathbb{X}^{\infty}_{q_+}, \qquad \bigcup_{\substack{t\in J_0,\\ u\in\mathbb{R}}} \big(u,\, F(t,u)\big) \subseteq {}^{2m_-}_{2M_-}\mathbb{X}^{2M_+}_{2m_+}\,. \tag{3.3.3}$$

The assumptions (3.3.2) and (3.3.3) can be written alternately in the form

$$q_\pm > 0, \quad -\sqrt{q_+} < m_+ \leq M_+ < \sqrt{q_+}, \quad -\sqrt{q_-} < m_- \leq M_- < \sqrt{q_-}, \tag{3.3.4}$$

$$\tilde{g}(t,u) \geq q_+ u, \quad t \in J_0,\ u \geq 0, \qquad \tilde{g}(t,u) \leq q_- u, \quad t \in J_0,\ u \leq 0, \tag{3.3.5}$$

$$2(m_+ u^+ - M_- u^-) \;\leq\; F(t,u) \;\leq\; 2(M_+ u^+ - m_- u^-), \quad t \in J_0,\ u \in \mathbb{R}. \tag{3.3.6}$$

Incidentally, if for $q > 0$ it holds $u\tilde{g}(t,u) \geq qu^2$, $t \in J_0$, $u \in \mathbb{R}$, then the graph of the function $u \mapsto \tilde{g}(t,u)$ is contained in ${}^{q}_{\infty}\mathbb{X}^{\infty}_{q}$ uniformly with respect to $t \in J_0$, i. e. (3.3.5) is satisfied with $q_+ = q_- = q$.

3.4. Uniformly oscillatory equation

Eq. (3.2.1) is said to be *uniformly oscillatory (about zero at* $+\infty$*)* if there exists (the so-called uniform) oscillatory time θ such that any non-zero $u \in \mathcal{U}$ is oscillatory with the same oscillatory time θ.

Theorem 3.1. *Under the above assumptions Eq.* (3.2.1) *is uniformly oscillatory and the uniform oscillatory time is given by*

$$\theta = \max \left\{ \vartheta^{q_+}_{-M_+} + \vartheta^{q_+}_{m_+},\ \vartheta^{q_-}_{-M_-} + \vartheta^{q_-}_{m_-} \right\}. \tag{3.4.1}$$

Sketch of the proof. Let $J \subset J_0$, $|J| > \vartheta^{q_\pm}_{-M_\pm} + \vartheta^{q_\pm}_{m_\pm}$, and u is of fixed sign in J, that is, either $u \geq 0$ (or $u \leq 0$) in J. In the former case $\operatorname{sgn} u = +1$, in the latter case $\operatorname{sgn} u = -1$ and the lower indices at q, m and M are $+$ or $-$ in accordance with the sign of u. We prove that $u = 0$ in J_0.

By assumption (3.3.2) and since the function $(q,\, p) \mapsto \vartheta^q_p$ is monotonically decreasing in q while p is fixed, there exists $\varepsilon > 0$ such that

$$(q_\pm - \varepsilon,\, -M_\pm)\,,\ (q_\pm - \varepsilon,\, m_\pm) \in \mathcal{O}$$

and

$$|J| \geq \vartheta^{q_\pm - \varepsilon}_{-M_\pm} + \vartheta^{q_\pm - \varepsilon}_{m_\pm} > \vartheta^{q_\pm}_{-M_\pm} + \vartheta^{q_\pm}_{m_\pm}.$$

Now, let us take any subinterval of the interval J of the length $\vartheta^{q_\pm - \varepsilon}_{-M_\pm} + \vartheta^{q_\pm - \varepsilon}_{m_\pm}$ and denote its end points by τ_1 and τ_2. We define the test function in the form

$$\gamma(t) = C(t - \tau_1, q_\pm - \varepsilon, -M_\pm, m_\pm) \tag{3.4.2}$$

where C is the universal comparison function (2.2.1) the properties of which are stated in Sec. 2.2. In particular,

$$\ddot{\gamma} + 2\left(-M_\pm \dot{\gamma}^+ + m_\pm \dot{\gamma}^-\right) + (q_\pm - \varepsilon)\,\gamma = 0.$$

Multiplying equation (3.2.1) by γ and integrating over the interval $(\tau_1,\, \tau_2)$ we get

$$\begin{aligned} 0 = \dot{\gamma}(\tau_1)\, u(\tau_1) - \dot{\gamma}(\tau_2)\, u(\tau_2) + \int_{\tau_1}^{\tau_2} [\ddot{\gamma} u - F(t,\, u)(\dot{\gamma}^+ - \dot{\gamma}^-) + \tilde{g}(t, u)\, \gamma]\, dt \geq \\ \geq \operatorname{sgn} u \int_{\tau_1}^{\tau_2} \left[\ddot{\gamma} + 2\left(-M_\pm \dot{\gamma}^+ + m_\pm \dot{\gamma}^-\right) + q_\pm\, \gamma\right]\, u\, dt = \varepsilon \operatorname{sgn} u \int_{\tau_1}^{\tau_2} \gamma\, u\, dt. \end{aligned}$$

Since $\varepsilon > 0$ and $\gamma > 0$ in $(\tau_1,\, \tau_2)$ we obtain $|u| \leq 0$ in $(\tau_1,\, \tau_2)$ and consequently $u = 0$ in J since $(\tau_1,\, \tau_2)$ is arbitrary. But this means that $u = 0$ in J_0. The proof is complete.

3.5. Examples

The present theory can be applied to equations of form (3.2.1) with the functions f and g independent of t, the so-called generalized Liénard equations,

$$\ddot{u} + f(u)\,\dot{u} + g(u) = 0, \qquad t \in J_0. \tag{3.5.1}$$

Both important cases of the behaviour of the damping term are covered: in the first, $f(u) \geq 0$ for all u, in the second $f(u) < 0$ for $|u|$ small.

Example 3.5.1.
Let $f,\, g \in C^1(\mathbb{R})$. By LaSalle's invariance principle the assumptions $f(u) > 0$, $u \in \mathbb{R}$, $u\,g(u) > 0$, $u \in \mathbb{R} \setminus \{0\}$, $G(u) := \int_0^u g(\bar{u})\,d\bar{u} \to +\infty$ for $|u| \to +\infty$ are sufficient for the zero solution of Eq. (3.5.1) to be globally asymptotically stable. Imposing more stringent assumptions on g and f, namely, $u\,g(u) \geq q\,u^2$, $u \in \mathbb{R}$, for some $q > 0$, and $0 < f(u) \leq 2M$, $u \in \mathbb{R}$, (this implies (3.3.5) with $q_- = q_+ = q$ and (3.3.6) with $M_+ = M_- = M$, $m_+ = m_- = 0$) the above theory yields the convergence of solutions to the zero equilibrium position in an oscillatory manner and the uniform oscillatory time is $\theta = \vartheta^q_{-M} + \vartheta^q_0$.

Example 3.5.2.
The theory can be applied also to Eq. (3.5.1) with negative damping, for example, $f(u) = -\cos\frac{\pi u}{\delta}$, $u \in (-\delta,\, \delta)$ for some $\delta > 0$, $f(u) = 1$ otherwise. More generally, let for some m and M, $2m \leq f(u) \leq 2M$, $u \in \mathbb{R}$, (this means that (3.3.6) is fulfilled with $M_+ = M_- = M$, $m_+ = m_- = m$). Then, if $u\,g(u) \geq q\,u^2$, $u \in \mathbb{R}$, for some $q > 0$, we get that all solutions are oscillatory (in agreement with [8] where $f(0) < 0$ and the sign condition $u\,F(u) > 0$ for $|u| \geq \delta_0 > 0$ is assumed) and the uniform oscillatory time is given by $\theta = \vartheta^q_{-M} + \vartheta^q_m$.

4. Partial differential equations

Partial differential equations occur in the study of models describing the so-called systems with distributed parameters. A number of such systems which are conservative similarly as the harmonic oscillator in Example 1.2.1 is described by the wave equation. Via this wave equation we introduce the concepts of globally oscillatory solution and uniformly globally oscillatory equation (in accordance with [2], [9], [15]). "Perhaps the most notable disadvantage associated with conservative systems is the fact that they do not occur in nature" ([4], p. 433). The presence of dissipation mechanisms in systems make the study of oscillatory solutions much more difficult. Examples of such dissipative terms in equations will be seen in Secs. 4.4, 4.5 and 5.6. We formulate conditions for some equations with dissipative terms to be uniformly globally oscillatory. We start with an auxiliary result from the theory of elliptic equations.

4.1. An eigenvalue problem

The following result on the eigenvalue problem for the Laplace operator Δ on a bounded domain $\Omega \subset \mathbb{R}^N$ with the homogeneous Dirichlet boundary condition is well-known:

$$\textit{there exists } \lambda_1 > 0 \quad \textit{and} \quad v_1 \in \overset{\circ}{W}{}_2^1(\Omega) \cap C^\infty(\Omega) \ \textit{such that}$$
$$-\Delta v_1 = \lambda_1 v_1 \quad \textit{and} \quad v_1 > 0 \ \textit{in } \Omega.$$

Moreover, λ_1 is a simple eigenvalue and the smallest one, $\lambda_1 = \inf\limits_{0 \neq v \in \overset{\circ}{W}{}_2^1(\Omega)} \dfrac{|\nabla v|^2}{|v|^2}$, where $|\cdot|$ means the norm in $L_2(\Omega)$. If Ω has a sufficiently regular boundary $\partial\Omega$ then $v_1 \in C^\infty(\bar{\Omega})$.
Example. For $N = 1$ and $\Omega = (0, \pi)$ we have $\lambda_1 = 1$ and $v_1 = \sin x$.

4.2. Conservative systems

The wave equation in a bounded domain $\Omega \subset \mathbb{R}^N$ for a function $(t, x) \mapsto u(t, x)$, $u\colon J_0 \times \Omega \to \mathbb{R}$, $J_0 = \mathbb{R}$, supplemented with the homogeneous Dirichlet boundary condition, namely,

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \Delta u(t,x) = 0, \quad (t,x) \in J_0 \times \Omega, \tag{4.2.1}$$

$$u(t,x) = 0, \quad (t,x) \in J_0 \times \partial\Omega, \tag{4.2.2}$$

has a solution, determined by the first frequency-mode pair (λ_1, v_1),

$$(t,x) \mapsto \sin(\sqrt{\lambda_1} t)\, v_1(x). \tag{4.2.3}$$

If we denote

$$\theta = \frac{\pi}{\sqrt{\lambda_1}} \tag{4.2.4}$$

then the function (4.2.3) is non-negative in $[0, \theta] \times \bar{\Omega}$ and positive in $(0, \theta) \times \Omega$. We prove (by [2]) that in fact *any solution cannot remain non-negative nor non-positive (almost) everywhere in Ω for a time period $J \subset J_0$ longer than θ unless it is a zero solution.* For u continuous this can be said in other words: for any interval $J \subset J_0$, $|J| > \theta$, there exist couples (t_1, x_1), $(t_2, x_2) \in J \times \Omega$ such that $u(t_1, x_1) < 0 < u(t_2, x_2)$, and, consequently, $u(t_0, x_0) = 0$ for some $(t_0, x_0) \in J \times \Omega$.

To be more explicit, by a solution of the problem (4.2.1) – (4.2.2) we mean the energy solution, this function belongs to the space $C(\mathbb{R}; \overset{\circ}{W}{}_2^1(\Omega)) \cap C^1(\mathbb{R}; L_2(\Omega))$ and turns out to be uniquely determined if we add the initial conditions (at $t_0 \in \mathbb{R}$ an arbitrarily given time)

$$u(t_0, x) = u_0(x), \qquad \frac{\partial u(t_0, x)}{\partial t} = u_1(x), \quad x \in \Omega, \tag{4.2.5}$$

where $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$, $u_1 \in L_2(\Omega)$.

Sketch of the proof. Let $J = [\tau_1, \tau_2]$, $|J| = \tau_2 - \tau_1 > \theta$ and $u \geq 0$ (a. e.) in $J \times \Omega$. We prove that $u = 0$ (a. e.) in $J_0 \times \Omega$. Choose $\varepsilon > 0$ such that $|J| = \pi/\sqrt{\lambda_1 - \varepsilon}$. Setting

$$\gamma(t) := \frac{1}{\sqrt{\lambda_1 - \varepsilon}} \sin(\sqrt{\lambda_1 - \varepsilon}\,(t - \tau_1)) \tag{4.2.6}$$

we obtain a function with the following properties:

- $\gamma \in C^\infty([\tau_1, \tau_2])$,
- $\gamma > 0$ in (τ_1, τ_2), $\quad \gamma(\tau_1) = \gamma(\tau_2) = 0$,
- $\dot\gamma(\tau_1) = 1$, $\ \dot\gamma(\tau_2) = -1$,
- $\ddot\gamma + (\lambda_1 - \varepsilon)\,\gamma = 0$ in (τ_1, τ_2).

Choosing the test function

$$z(t, x) = \gamma(t)\, v_1(x) \tag{4.2.7}$$

we get

$$0 = \dot\gamma(\tau_1) \int_\Omega u(\tau_1, x)\, v_1(x)\, dx - \dot\gamma(\tau_2) \int_\Omega u(\tau_2, x)\, v_1(x)\, dx +$$

$$+ \int_{\tau_1}^{\tau_2} \int_\Omega (\ddot\gamma + \lambda_1 \gamma)\, u(t, x)\, v_1(x)\, dxdt \geq \varepsilon \int_{\tau_1}^{\tau_2} \int_\Omega \gamma(t)\, u(t, x)\, v_1(x)\, dxdt.$$

Since $\varepsilon > 0$, $\gamma > 0$ in (τ_1, τ_2) and $v_1 > 0$ in Ω we have $u \equiv 0$ in $(\tau_1, \tau_2) \times \Omega$. By virtue of the unique solvability of the initial boundary value problem we get $u \equiv 0$ in $\mathbb{R} \times \Omega$.

The same conclusion is obtained if we assume $u \leq 0$ on $J \times \Omega$ where $J \subset J_0$ is any interval the length $|J|$ of which is greater than (4.2.4).

Remark 4.2.1.
This *global* oscillation result is optimal in the sense that (4.2.4) is the smallest of values with such a property. Indeed, the function (4.2.3) is a solution that is *positive* on $(0, \pi/\sqrt{\lambda_1}) \times \Omega$. We are led to the following definitions.

4.3. Globally oscillatory function

A (Lebesgue) measurable function

$$(t, x) \mapsto u(t, x), \quad u\colon J_0 \times \Omega \to \mathbb{R}, \tag{4.3.1}$$

($J_0 = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$ or $J_0 = \mathbb{R}$), is said to be *globally oscillatory (about zero at $+\infty$)* if there exists (the so-called *oscillatory time*) $\theta > 0$ such that for any interval $J \subset J_0$, $|J| > \theta$, we have simultaneously

$$\operatorname{meas}\{\,(t, x) \in J \times \Omega \mid u(t, x) > 0\,\} > 0 \quad \text{and} \quad \operatorname{meas}\{\,(t, x) \in J \times \Omega \mid u(t, x) < 0\,\} > 0.$$

Provided u possesses the (pseudo-analyticity) property: *if* $u = 0$ *a. e. in* $J \times \Omega$*,* $J \subset J_0$ *a compact (non-degenerate) interval, then* $u = 0$ *a. e. in* $J_0 \times \Omega$, an equivalent definition is possible: a measurable function $u\colon J_0 \times \Omega \to \mathbb{R}$, $u \neq 0$ a. e. in $J_0 \times \Omega$, is globally oscillatory if and only if there exists $\theta > 0$ such that for any interval $J \subset J_0$ the following implication holds:

$$u \geq 0 \;\; (u \leq 0, \;\textit{respectively}) \;\; \textit{a. e. in } J \times \Omega \implies |J| \leq \theta. \tag{4.3.2}$$

Roughly speaking, this means, for a continuous function $u \, (\not\equiv 0)$, that u has a zero in any domain $J \times \Omega$ where $J \subset J_0$ is an interval the length of which is sufficiently large and this length can be chosen independently of J.

The notion of oscillatory time replies to the question: *for how long time period at most can a non-zero function remain non-negative (non-positive) in the domain* Ω*?*

A partial differential equation (for a function (4.3.1)) supplemented with a boundary condition is said to be *uniformly globally oscillatory (about zero at* $+\infty$) if there exists (the so-called *uniform*) oscillatory time θ such that any non-zero solution (of a relevant class, e. g. energy solution) is globally oscillatory with the same oscillatory time θ.

The results of Sec. 4.2 can be restated in the following manner. *The wave equation* (4.2.1) *(supplemented with the boundary condition* (4.2.2)*) is uniformly globally oscillatory and the uniform oscillatory time is given by* (4.2.4).

Remark 4.3.1.
Furthermore, all non-zero (sufficiently regular) solutions are not only globally oscillatory but also "locally oscillatory". This is a consequence of the fact that for a fixed $x_0 \in \Omega$ the function $t \mapsto u(t, x^0)$ is almost periodic, for $n = 1$ even periodic, with the mean value zero.

4.4. Dissipative systems with internal damping

The study of oscillatory properties of equations with dissipative terms is more delicate. We present the results for the wave equation with two kinds of dampings (following [13]). The proofs rely on the special choice of the test function in the form (4.2.7) where v_1 has the same meaning as in the Sec. 4.1 and the function (4.2.6) has to be substituted by the function (3.4.2). The estimates of dissipative terms are performed similarly as it was done in the proof of Theorem 3.1.

Let us consider the equation for a function $(t, x) \mapsto u(t, x)$, $u\colon J_0 \times \Omega \to \mathbb{R}$, $J_0 = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently regular boundary $\partial\Omega$ and $N = 1, 2$ or 3:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \sigma(u) + 2a\frac{\partial u}{\partial t} - 2b\,\Delta\frac{\partial u}{\partial t} = 0, \quad (t, x) \in J_0 \times \Omega, \tag{4.4.1}$$

supplemented with the homogeneous Dirichlet boundary condition (4.2.2). Here a and b are constants, $a \geq 0$, $b > 0$ (the case $b = 0$ is dealt with in the next section) and $u \mapsto \sigma(u)$ is a function. Equation (4.4.1) has been studied by a number of authors (see, e. g. [1] and [11] for other bibliographical references). The interest in this equation is due to the fact that it governs the motion of a damped linear visco-elastic (Kelvin) solid (a bar if $N = 1$, a plate if $N = 2$) subjected to nonlinear elastic constraints.

Let us assume further: $\sigma \in C^1(\mathbb{R})$, $\sigma(0) = 0$ and there exists $\sigma_0 > -\lambda_1$ (where λ_1 is the smallest eigenvalue of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, see Sec. 4.1) such that $\sigma'(u) \geq \sigma_0$, $u \in \mathbb{R}$.

The initial boundary value problem given by (4.4.1), (4.2.2) and by the initial conditions (4.2.5) has for any $u_0 \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and $u_1 \in L_2(\Omega)$ a unique global energy solution. Moreover,

$$u \in L_\infty\big(J_0; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)\big) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L_\infty\big(J_0; L_2(\Omega)\big) \cap L_2\big(J_0; \overset{\circ}{W}{}_2^1(\Omega)\big),$$

and there exist $M > 0$ and $\beta > 0$ (independently of u) such that

$$\left|\frac{\partial u(t,\cdot)}{\partial t}\right|^2 + |\Delta u(t,\cdot)|^2 \leq M\, e^{-\beta t}, \quad t \in J_0,$$

where $|\cdot|$ means the norm in $L_2(\Omega)$. In fact, this solution decays to 0 for $t \to +\infty$ in an oscillatory way if a (≥ 0) and b (> 0) are sufficiently small. Moreover, the existence of the uniform oscillatory time is ensured.

Theorem 4.1. *Under the above assumptions and if*

$$a \geq 0, \quad b > 0, \quad a + \lambda_1 b < \sqrt{\lambda_1 + \sigma_0}, \tag{4.4.2}$$

then Eq. (4.4.1) *(subject to* (4.2.2)*) is uniformly globally oscillatory and the uniform oscillatory time is given by*

$$\theta = \vartheta^{\lambda_1+\sigma_0}_{-a-\lambda_1 b} + \vartheta^{\lambda_1+\sigma_0}_{0}.$$

4.5. Dissipative systems with viscous damping

For $b = 0$ Eq. (4.4.1) takes the form

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \sigma(u) + 2a\frac{\partial u}{\partial t} = 0, \quad (t,x) \in J_0 \times \Omega. \tag{4.5.1}$$

Let $a > 0$, $\sigma \in C^1(\mathbb{R})$. Denote $\Xi(u) = \int_0^u \sigma(\bar{u})\, d\bar{u}$. Let there exist constants $C_0 \geq 0$, $C_1 \geq 0$, $C_2 \geq 0$, $\delta > 0$ such that $\Xi(u) \geq -C_0$, $\delta\, \Xi(u) \leq C_1 + \sigma(u)\, u$, $u \in \mathbb{R}$. Let either $N = 1$ or σ' satisfy the growth condition $|\sigma'(u)| \leq C_2\,(1 + |u|^\gamma)$, $u \in \mathbb{R}$, where $0 \leq \gamma < +\infty$ if $N = 2$, otherwise $0 \leq \gamma < 2/(N-2)$.

For any $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$ and $u_1 \in L_2(\Omega)$ the initial boundary value problem given by (4.5.1), (4.2.2) and by the initial conditions (4.2.5) has a unique global bounded energy solution

$$u \in L_\infty\big(J_0; \overset{\circ}{W}{}_2^1(\Omega)\big) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L_\infty\big(J_0; L_2(\Omega)\big).$$

Assumptions ensuring oscillations: let there exist a constant σ_0 such that

$$\lambda_1 + \sigma_0 > 0, \quad a < \sqrt{\lambda_1 + \sigma_0}$$

(for λ_1 see Sec. 4.1) and $u\,\sigma(u) \geq \sigma_0\, u^2$, $u \in \mathbb{R}$. A function $\sigma \in C^1(\mathbb{R})$ fulfils the latter assumption if, for example, σ is odd, $\sigma(u) > \sigma_0 u$, $u > 0$.

Theorem 4.2. *Under the above assumptions Eq.* (4.5.1) *(subject to* (4.2.2)*) is uniformly globally oscillatory and the uniform oscillatory time is given by*

$$\theta = \frac{\pi}{\sqrt{\lambda_1 + \sigma_0 - a^2}} \qquad \big(= \vartheta_{-a}^{\lambda_1+\sigma_0} + \vartheta_{a}^{\lambda_1+\sigma_0}\big)\,.$$

5. Abstract differential equations

Many differential operators exhibit similar properties as the operator $-\Delta$ operating on functions defined in a bounded domain in $\mathbb{R}^N$ and vanishing on its boundary, namely the positivity of the first eigenvalue and the positivity of the corresponding eigenfunction. Analogous suitable properties are encountered with a number of abstract operators in ordered Banach spaces (see e. g. [23]). Moreover, it is possible to define the notion of oscillation for abstract-valued functions with values in abstract ordered Banach spaces (non-negativity of a function is replaced by its appearance in the cone defined by the ordering). As an example, we investigate oscillatory properties of an abstract evolution equation that can be used for modelling some general elastic systems. A suitable linear operator in an abstract Hilbert space determines its linear (elasticity) part and some linear dissipative terms together with a nonlinear term are included. An application of obtained results to Euler–Bernoulli beam model is shown in the end.

5.1. Spaces

Let V and H be (real) Hilbert spaces, V be continuously and densely embedded in H, i. e. $V \hookrightarrow H$, V dense in H. Identifying H and its dual H' we get $V \hookrightarrow H \hookrightarrow V'$ and both embeddings are continuous and dense. The duality between V' and V is denoted by the same symbol $\langle \cdot, \cdot \rangle$ as the scalar product in H. The norms in H and V, respectively, are denoted by $|\cdot|$ and $\|\cdot\|$, respectively.

Let K be a cone (more specifically, a pointed convex cone of vertex 0) in the space V (i. e. a nonempty set satisfying: (i) $u, v \in K$, $\alpha, \beta > 0 \implies \alpha u + \beta v \in K$, and (ii)

$u, -u \in K \implies u = 0$). The cone K induces an ordering $\preceq$ in the space V by writing $u \preceq v \iff v - u \in K$ and V becomes an ordered space with the cone K of non-negative elements. (The converse is also true: in any linear ordered space the ordering defines a cone $K = \{ v \in V \mid 0 \preceq v \}$.) For a closed cone K a dual description is available: $K = \{ v \in V \mid \langle w, v \rangle \geq 0 \text{ for all } w \in K' \}$ where K' is the set (wedge) of all non-negative linear continuous functionals on V, $K' = \{ w \in V' \mid \langle w, v \rangle \geq 0 \text{ for all } v \in K \}$, in the dual space V' (see [23] where also conditions for K' to be a cone, the so-called dual cone with respect to the cone K, are given). The d-interior of K (the set of the so-called quasi-interior points) is defined by $K^d = \{ v \in K \mid \langle w, v \rangle > 0 \text{ for all } w \in K' \setminus \{0\} \}$. If K contains some ball of positive radius (K is called solid) then the interior of K is equal to the d-interior, $\operatorname{Int} K = K^d$, but in some important applications (to partial differential equations) K^d is not empty in contrast to $\operatorname{Int} K$. *Example.* In the space $L_2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable domain, the set $K = \{ v \in L_2(\Omega) \mid v(s) \geq 0 \text{ almost everywhere in } \Omega \}$ is a cone. In this case $K' = K$ and $\emptyset = \operatorname{Int} K \neq K^d = \{ v \in L_2(\Omega) \mid v(s) > 0 \text{ a. e. in } \Omega \}$.

5.2. Operators

Let $((\cdot, \cdot))$ be a continuous bilinear form on $V \times V$ that is symmetric and V–elliptic (there exists $c_0 > 0$ with $((u, u)) \geq c_0 ||u||^2$, $u \in V$). The form $((\cdot, \cdot))$ determines a unique operator $L\colon V \to V'$ such that $((u, v)) = \langle Lu, v \rangle$, $u, v \in V$, and (by Lax-Milgram theorem) this operator is an isomorphism of V onto V'. If we define the domain of L in H by $D(L) = \{u \in V \mid Lu \in H\}$ we are entitled to consider L as a linear unbounded densely defined operator $L\colon D(L) \to H$, this operator being positive definite and self-adjoint along with its inverse L^{-1}. If $D(L)$ is equipped with the graph norm then L is also an isomorphism of $D(L)$ onto H.

Further, we assume that V is compactly embedded in H, consequently L^{-1} is a self-adjoint compact operator in H and elementary spectral theory of self-adjoint compact operators in a Hilbert space (of non-finite dimension) can be used. In particular: there exists a complete orthonormal set in H of eigenvectors $\{v_k\}_{k=1}^{\infty}$ of L with the corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$,

$$0 < \inf_{\substack{v \in V \\ v \neq 0}} \frac{\langle Lv, v \rangle}{|v|^2} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \quad \text{and } \lambda_k \to \infty \text{ as } k \to \infty.$$

For any self-adjoint positive definite unbounded operator L in H powers L^s can be defined for any $s \in \mathbb{R}$. Under the above assumption on compactness the powers L^s for $s > 0$ are characterized by $D(L^s) = \{v \in H; \sum_{k=1}^{\infty} \lambda_k^{2s} \langle v, v_k \rangle^2 < \infty\}$ and $L^s v = \sum_{k=1}^{\infty} \lambda_k^s \langle v, v_k \rangle v_k$. The operators L^s for $s > 0$ are unbounded densely defined operators in H that are self-adjoint and positive definite. For $s = \frac{1}{2}$ we have $D(L^{\frac{1}{2}}) = V$.

We adopt the fundamental assumption:

$$\textit{there exists } \lambda_1 > 0 \textit{ and } v_1 \in K^d \textit{ such that } L v_1 = \lambda_1 v_1. \tag{5.2.1}$$

As a consequence of the fact that v_1 belongs to the d-interior K^d of the cone K we have the following implication:

$$w \in K',\ \langle w,\, v_1\rangle \le 0 \Longrightarrow w = 0. \tag{5.2.2}$$

The assumption $K \subset K'$ enables us to use (5.2.2) for $w \in K$.

5.3. Abstract equation and energy solutions

We deal with the equation for the function $t \mapsto u(t)$, $u\colon J_0 \to V$, $J_0 = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$:

$$\ddot{u} + \big(a_1\, L^{\frac{1}{2}} + a_2\, L\big)u + \sigma(t,u) + \ 2\big(\alpha_0 + \alpha_1\, L^{\frac{1}{2}} + \alpha_2\, L\big)\dot{u} = 0, \quad t \in J_0, \tag{5.3.1}$$

where $a_1, a_2, \alpha_0, \alpha_1, \alpha_2$ are constants and σ is an operator, $(t,u) \mapsto \sigma(t,u)$, $\sigma\colon J_0 \times V \to V'$, $\sigma(t,0) \equiv 0$, $t \in J_0$, satisfying suitable continuity, measurability and growth conditions.

Since we are interested in oscillatory properties of global solutions, i. e. defined for $t \in J_0$, we adopt the assumption of existence and uniqueness of global energy solutions of the associated initial value problem.

By a global *energy solution* of Eq. (5.3.1) we mean a function $u \in C(J_0; V) \cap C^1(J_0; H) \cap C^2(J_0; V')$ such that for any compact interval $J \subset J_0$ the equality

$$\int_J \Big[\langle -\dot{u} - 2\big(\alpha_0 + \alpha_1\, L^{\frac{1}{2}} + \alpha_2\, L\big)\, u,\ \dot{z}(t)\rangle + \langle \big(a_1\, L^{\frac{1}{2}} + a_2\, L\big)u + \sigma(t,u),\ z(t)\rangle\Big]\, dt = 0$$

holds for any $z \in \overset{\circ}{W}{}^1_2(J;\, V)$ (the test function z belongs to this space if and only if $z\colon J \to V$ is an absolutely continuous function, z vanishes at boundary points of J, the (strong) derivative $\dot{z}$ exists almost everywhere on J, and $\dot{z} \in L_2(J; V)$).

5.4. *K*-oscillatory functions

In the framework of ordered spaces an abstract oscillatory function can be defined (cf. [3], [10], [15]). A (Lebesgue) measurable function $t \mapsto u(t)$, $u\colon J_0 \to V$, is said to be K-*oscillatory (about zero at* $+\infty$*)* if there exists (the so-called *oscillatory time*) $\theta > 0$ such that for any interval $J \subset J_0$, $|J| > \theta$, we have simultaneously

$$\operatorname{meas}\{\, t \in J \mid u(t) \in K \setminus \{\, 0\,\}\} > 0 \quad \text{and} \quad \operatorname{meas}\{\, t \in J \mid -u(t) \in K \setminus \{\, 0\,\}\} > 0.$$

Provided u possesses the (pseudo-analyticity) property: *$u = 0$ a. e. in $J \subset J_0$, J compact (non-degenerate) interval, then $u = 0$ a. e. in J_0*, the definition can be said in different words. A measurable function $u\colon J_0 \to V$, $u \ne 0$ a. e. in J_0, is K-oscillatory if and only if there exists $\theta > 0$ such that for any interval $J \subset J_0$ the following implication holds:

$$u(t) \in K \ \ (-u(t) \in K,\ \textit{respectively}) \ \ \textit{a. e. in} \ \ J \Longrightarrow |J| \le \theta.$$

5.5. Uniformly K-oscillatory equation

Eq. (5.3.1) is called *uniformly* K*-oscillatory (about zero at* $+\infty$) if there exists (the so-called *uniform*) oscillatory time θ such that any non-zero energy solution is K-oscillatory with the same oscillatory time θ.

Let $a_1,\ a_2,\ \alpha_0,\ \alpha_1,\ \alpha_2, \sigma_+,\ \sigma_- \in \mathbb{R}$, $a_2 > 0$. Denote

$$\alpha = \alpha_0 + \alpha_1\sqrt{\lambda_1} + \alpha_2\lambda_1, \qquad E = \min\{\sigma_+,\ \sigma_-\} + a_1\sqrt{\lambda_1} + a_2\lambda_1 \tag{5.5.1}$$

and assume

$$E > 0, \quad |\alpha| < \sqrt{E}. \tag{5.5.2}$$

Further, let

$$\langle\sigma(t,u), v_1\rangle \geq \sigma_+\langle u, v_1\rangle, \quad t \in J_0,\ u \in K \setminus \{\,0\,\}, \tag{5.5.3}$$

$$\langle\sigma(t,u), v_1\rangle \leq \sigma_-\langle u, v_1\rangle, \quad t \in J_0,\ -u \in K \setminus \{\,0\,\}. \tag{5.5.4}$$

Theorem 5.1. *Under the above assumptions any energy solution* u *of Eq.* (5.3.1) *can neither remain in* K *nor* $-K$ *for the time period* $J \subset J_0$ *greater than*

$$\theta = \vartheta^E_{-|\alpha|} + \vartheta^E_0, \tag{5.5.5}$$

unless it is identically equal to 0. In other words, *Eq.* (5.3.1) *is uniformly* K*-oscillatory about the zero equilibrium position with the uniform oscillatory time* θ *given by* (5.5.5).

5.6. Application

As an example of the application of the theory we examine the classical Euler–Bernoulli beam model under the presence of damping terms (see [18], [19]).

Let $\Omega = I = (0,\ \ell)$, $H = L_2(I)$, $V = W_2^2(I) \cap \overset{\circ}{W}{}_2^1(I)$, K be the cone of non-negative a. e. in I functions from V. Let $(\!(u,\ v)\!) = \langle u'',\ v''\rangle$, $u,\ v \in V$, hence

$$Lv = v'''', \qquad v(0) = v''(0) = 0, \quad v(\ell) = v''(\ell) = 0, \tag{5.6.1}$$

i. e. $D(L) = \{\, v \in W_2^4(I) \cap \overset{\circ}{W}{}_2^1(I),\ v'' \in \overset{\circ}{W}{}_2^1(I)\,\}$.

We use here the fact that the square root of the operator (5.6.1) is a *differential* operator

$$L^{\frac{1}{2}}v = -v'', \qquad v(0) = v(\ell) = 0, \tag{5.6.2}$$

i. e. $D(L^{\frac{1}{2}}) = W_2^2(I) \cap \overset{\circ}{W}{}_2^1(I)$. For other boundary conditions, in general, there is a difference between the square root $L^{\frac{1}{2}}$ of the fourth-order derivative operator and the negative second derivative, by [25]: $-v'' = [Id. + P]\, L^{\frac{1}{2}} v$, where P is a bounded, but in general not compact, operator in $L_2(I)$.

In this case $Lv_1 = \lambda_1 v_1$ where

$$\lambda_1 = \left(\frac{\pi}{\ell}\right)^4 \quad \text{and} \quad v_1 = \sin\frac{\pi x}{\ell}. \tag{5.6.3}$$

Let a_1, a_2, α_0, α_1, α_2 be constants, $(t,x,u) \mapsto g(t,x,u)$, $g\colon J_0 \times I \times \mathbb{R}$ (satisfying suitable continuity and growth conditions). Then, with the notation $\sigma(t,u) = g(t,x,u)$, in fact, Eq. (5.3.1) becomes a partial differential equation for a function $(t,x) \mapsto u(t,x)$, $u\colon J_0 \times I \to \mathbb{R}$:

$$\frac{\partial^2 u}{\partial t^2} + a_2 \frac{\partial^4 u}{\partial x^4} - a_1 \frac{\partial^2 u}{\partial x^2} + g(t,x,u) + \; 2\left(\alpha_0 \frac{\partial u}{\partial t} - \alpha_1 \frac{\partial^3 u}{\partial t\, \partial x^2} + \alpha_2 \frac{\partial^5 u}{\partial t\, \partial x^4}\right) = 0. \tag{5.6.4}$$

complemented by the boundary conditions

$$u(0,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = 0, \qquad u(\ell,t) = \frac{\partial^2 u(\ell,t)}{\partial x^2} = 0, \qquad t \in J_0. \tag{5.6.5}$$

Assuming, for instance, $ug(t,x,u) \geq \sigma_0 u^2$, $t \in J_0$, $x \in I$, $u \in \mathbb{R}$, and setting $\sigma_+ = \sigma_- = \sigma_0$, the results of the previous section can be used.

If α_0, α_1 and α_2 are vanishing then Eq. (5.6.4) represents (see [27]) the classical Euler–Bernoulli model for transverse (lateral) vibrations of a simply supported beam which is modified by terms representing a nonlinear elastic foundation on which the beam rests (the term $g(t,x,u)$) and an axial (tensile or compressive) force (the term $a_1 L^{\frac{1}{2}}$).

The terms

$$(a) \quad 2\alpha_0 \frac{\partial u}{\partial t}, \qquad (b) \quad -2\alpha_1 \frac{\partial^3 u}{\partial t\, \partial^2 x}, \qquad (c) \quad 2\alpha_2 \frac{\partial^5 u}{\partial t\, \partial^4 x} \tag{5.6.6}$$

with non-zero (small positive) α_0, α_1 and/or α_2 represent dampings. The term (a) introduces the so-called external or *viscous* damping. The amplitudes of all normal modes of the vibration are attenuated at the same rate (contrary to experience).

In [4] Chen and Russell proposed the so-called "square root" ($L^{\frac{1}{2}}$) model for which the so-called *structural* damping is achieved: damping rates of the normal modes are linearly proportional to the frequencies of the modes (see also [5], [26]). The term $2\alpha_1 L^{\frac{1}{2}} \frac{\partial u}{\partial t}$ is equal just to (b) in (5.6.6) (and, moreover, with very natural interpretation that the damping force is proportional to the bending rate).

The presence of an additional term (c) in (5.6.4) means that the damping rates of the normal modes of vibration depend quadratically on frequency at low frequencies (consistent with empirical studies for some materials), high frequency modes are overdamped (and difficult to observe at all). This type of damping is the so-called *internal* or *Kelvin-Voigt* damping.

6. Conclusion and further goals

"Oscillation theory of differential equations, originated from the monumental paper of C. Sturm published in 1836, has now been recognized as an important branch of mathematical analysis from both theoretical and practical viewpoints" ([28]). There is a vast literature for oscillatory solutions of ordinary differential equations (see e. g. [24]). Substantially less is known on oscillatory solutions of partial evolution equations (see e. g. [28]), especially under the presence of dissipative terms. The solutions of partial differential equations are functions $(t, x) \mapsto u(t, x)$ of both time variable $t \in [t_0, +\infty)$ and "spatial" variable $x \in \Omega \subset \mathbb{R}^n$ and the notion of a globally oscillatory function comes into play (see [2], [3], [9], [10]). The oscillatory time – a quantitative measure of oscillation – defines *the maximal length of time intervals on which non-zero solutions u can remain non-negative (non-positive) in the domain Ω.* A basic tool for estimating the oscillatory time is the summit function defined in Sec. 2.1. Another result of basic importance, especially for estimating dissipative terms in equations, is the the existence of conjugate points for the ordinary second order differential equation with jumping nonlinearities in the first derivative. These facts are recalled in Sec. 2.2.

A further generalization is possible. The problem can be solved in abstract functional setting with the use of the theory of Banach spaces with a cone and linear operators reproducing a cone, the so-called non-negative operators. Let V be an ordered Banach space with the cone K of non-negative elements and let $G \colon V \to V$ be a linear operator. An operator G is called *(K-)non-negative* if $G(K) \subset K$, that is, $u \succeq 0 \implies G\,u \succeq 0$ (G leaves invariant the cone K). Instead of non-negative the term positive is used here often.

A rich and well-developed theory of non-negative operators yields additional assumptions under which there exist eigenvectors in the cone K and investigates other properties of these eigenvectors and corresponding eigenvalues (Kreĭn–Rutman type theorems). For instance, a wide class of operators is formed by the so-called superpositive operators whose spectral properties is described by the following definition. An operator G is called *superpositive* if there exists a positive eigenvalue λ_1, $\lambda_1 = \max\{\,|\lambda|; \lambda \in \text{spektrum } G\,\}$, λ_1 is simple, i. e. its multiplicity is equal to 1, to the eigenvalue λ_1 there is associated the corresponding eigenvector $v_1 \in K$ and λ_1 is the only eigenvalue corresponding to an eigenvector of the operator G in K (that is, $G\,v = \lambda\,v,\ v \in K \setminus \{\,0\,\} \Longrightarrow \lambda = \lambda_1$). The well-known Kreĭn–Rutman theorem yields the superpositivity of the operator G provided the cone K is closed, $\operatorname{Int} K \neq \emptyset$ and G is a linear compact operator satisfying the condition (of the so-called *strong positivity*) $G(K \setminus \{\,0\,\}) \subset \operatorname{Int} K$. This result has stimulated the investigation of spectral properties of non-negative operators without the assumption of non-voidness of the interior of K and with weakened assumptions on the compactness. The origin of the theory of non-negative operators dates back to Perron–Frobenius results on matrices: if $V = \mathbb{R}^N$ and K is the cone of vectors with non-negative coordinates then a square matrix G is superpositive provided all elements of G or of some iterate G^k are positive.

Concerning the linear continuous symmetric operator $L\colon V \to V'$ the presented oscillation theory relies on the assumption (5.2.1) together with its consequence (5.2.2).

This assumption is fulfilled, for instance, according to already mentioned Kreĭn–Rutman theorem, if V is compactly embedded in H, the cone K is closed, $\operatorname{Int} K \neq \emptyset$ and the operator L has the inverse L^{-1} satisfying $L^{-1}(K \setminus \{0\}) \subset \operatorname{Int} K$. Here λ_1 is the smallest eigenvalue of the operator L (which is known to be simple).

More interesting and important from the point of view of applications to partial differential equations are criteria without the assumption $\operatorname{Int} K \neq \emptyset$. Assumption (5.2.1) is fulfilled, for example, in the following case: $\Omega \subset \mathbb{R}^N$ bounded domain with a sufficiently regular boundary, $H = L_2(\Omega)$, $V = \overset{\circ}{W}{}_2^1(\Omega)$, $V' = W_2^{-1}(\Omega)$, $K = \{u \in V; u \geq 0 \text{ a. e. in } \Omega\}$, $c_0 \in C(\bar{\Omega})$, $C = \{c_{jk}\}_{j,k=1}^N$ is a matrix of functions from $C^1(\bar{\Omega})$ that is symmetric and positive definite uniformly with respect to $x \in \bar{\Omega}$ and $Lu = -\operatorname{div}\,(C(x)\operatorname{grad} u) + c_0(x)\,u$.

For Banach spaces with a cone we refer e. g. to [23].

The presented method enables to cope with equations involving more general nonlinear operators as well and this suggests further goals and the perspectives of research. For some results with nonlinearities see [15] and [17]. Various dissipation mechanisms yield new dissipative terms in equations and give rise to new subjects of study (for instance, global type external dissipation in the beam model is dealt with in e. g. [6]). Several systems exhibit the property that the oscillatory regime is settled only after a certain period of time had elapsed (see e. g. [6] and [29]). In this context the corresponding solution is said to be eventually oscillatory. Attention should be paid to further examination of such situations. A modification of the approach must be done if we investigate equations with time-dependent coefficients (for results in this direction see [21] and [22]). A further direction of development of the method is the study of oscillation properties of “scalar” functions $t \mapsto \langle w,\, u(t)\rangle$ for w from the dual space V' (as indicated in [20]).

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1966 – 1971 the Faculty of Mathematics and Physics of the Charles University in Prague

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Institute of Mathematics of the Czechoslovak Academy of Sciences, Prague,
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