České vysoké učení technické v Praze, Fakulta elektrotechnická

Czech Technical University in Prague, Faculty of Electrical Engineering

Doc. Ing. Pavel Zahradník, CSc.

Moderní trend v návrhu číslicových filtrů pro použití v telekomunikacích

Recent Trend in the Digital Filter Design for Telecommunication Applications

Summary

This document summarizes novel approach in the analytical design of narrow band finite impulse response (FIR) filters useful in telecommunication applications. I have published this topic in four papers [29] - [32] in the IEEE Transactions on Circuits and Systems in the period 2004-2007. Four types of the FIR filters are considered : the maximally flat notch FIR filter, the optimal equiripple notch FIR filter, the optimal equiripple DC-notch FIR filter and the optimal equiripple comb FIR filter. The analytical approach in the design of each particular filter type consists of three fundamental and new results : differential equation for the approximating polynomial of the filter, degree equation of the filter and an algorithm for the analytical evaluation of the impulse response of the filter. Each of the presented analytical design procedures starts with frequency specifications of the filter and ends at the set of the impulse response coefficients without any numerical recourse. Analytical design procedures produce filters with quantized critical frequencies. The presented fast analytical procedure for the tuning of FIR filters complements the analytical design by removing of this limitation. Selected applications of the narrow band FIR filters in telecommunication technology are mentioned.

Souhrn

V tomto dokumentu shrnuji nový přístup k analytickému návrhu úzkopásmových číslicových filtrů s konečnou impulsní odezvou (FIR) pro telekomunikační aplikace. Tuto problematiku jsem publikoval v čtyřech příspěvcích [29] - [32] v časopise IEEE Transactions on Circuits and Systems v letech 2004-2007. Uvažuji čtyři typy FIR filtrů : maximálně plochý notch FIR filtr, optimální notch FIR filtr s stejnoměrným zvlněním, optimální DC-notch FIR filtr s stejnoměrným zvlněním a optimální hřebenový FIR filtr s stejnoměrným zvlněním. Analytický přístup pro každý popisovaný typ filtru zahrnuje tři nové zásadní výsledky : diferenciální rovnici pro aproximační funkci filtru, rovnici pro stupeň filtru a rekurentní algoritmus pro výpočet koeficientů impulsní odezvy filtru. Analytický postup návrhu filtru vychází ze specifikace filtru, výsledkem návrhu jsou koeficienty impulsní odezvy filtru bez použití numerických postupů. Analytický návrh FIR filtrů vede k filtrům s kvantovanými polohami kritických kmitočtů. Navržená analytická metoda pro ladění FIR filtrů odstraňuje toto omezení. Zmiňuji vybrané aplikace úzkopásmových číslicových filtrů v telekomunikační technice.

Klíčová slova : číslicový FIR filtr, rychlý analytický návrh, maximálně plochý, stejnoměrné zvlnění, notch filtr, hřebenový filtr, ladění filtru, telekomunikace, aplikace

Key words : digital FIR filter, fast analytical design, maximally flat, equiripple, notch filter, comb filter, filter tuning, telecommunication, applications

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1. Preface

Digital filters represent basic functional block in the digital processing of telecommunication signals. The design of digital FIR filters looks like a closed chapter in history of digital signal processing. Reading several textbooks devoted to the DSP a reader is tempted to think that few numerical methods including Fast Fourier Transform and Remez exchange algorithm implemented by famous Parks-McClellan code [9], is all what is needed for the FIR filters. Recently only a limited number of closed form solutions is available. This document deals with analytical design of narrow band digital FIR filters for telecommunication applications including the class of the optimal filters which can not be obtained with Remez algorithm at all. My analytical approach to the FIR filters consists of three fundamental and new results :

1. differential equation for the approximating function
2. degree equation of the filter and
3. analytical algorithm for the coefficients of the impulse response.

The development of the differential equation is novel and its concept is essential. It consequently plays a fundamental role in replacing a spectral transformation by the algebraic evaluation of the impulse response coefficients. The filter degree is evaluated through the closed form formulae. It is a novel and non-standard result as the empirical estimation for the filter degree is frequently used [13], [14]. The solution of the differential equation provides the recurrence algorithm for the impulse response coefficients refraining from the Fast Fourier Transform seen indispensable in some attempts in the analytical filter design. If some analytical design methods are worthy of noting [3], it is to emphasize that there are always several numerical methods involved. In following the filter design starts with frequency specifications and ends at the set of the impulse response coefficients without any numerical recourse. On top of that, the presented analytical approach is extremely robust and fast. This properties are appreciated in the adaptive filtering.

2. Selected Applications of Narrow Band Digital Filters in Telecommunication Technology

Notch filter (Chap. 4 and 5) is a versatile type of filter which is generally used either for the attenuation of the signal or for the separation of a narrow band signal within some frequency band. Notch filters find numerous applications in telecommunication systems, wired, wireless, optical etc. They are used in the processing of base band signals, intermediate band signals and recently even in the digital processing of RF band signals. Some applications of the digital notch FIR filters include noise reduction in airborne communication containing high-amplitude peaks within the audio frequency range, processing of DTMF signals, frequency-difference detectors in phase-locked-loop, Costas-loop-based clock and carrier-recovery systems, detectors of useful signals and interference, interference attenuators, channel separators, local rejectors, protectors from the in or near-band interference, suppressors of mechanical resonance of MEMS micromirrors in optical switching and free space communication etc. Notch filters combined with the frequency tuning option (Chap. 6) are useful in the adaptive notch filtering. An example is the multiple carrier adaptive notch filter. It is used for ultra selective determination and elimination of the narrow band interference within the frequency spectrum of the wide band communication signal.

DC-notch (zero-frequency notch) filter (Chap. 7) is generally used to remove the bias in the signal. The DC-notch filters are frequently found in telecommunication systems. The DC compensation is often required in FM receivers. In the image and video communication, the DC notch filter is used to compensate for varying and unevenly distributed illumination in the image area resulting from the process of image acquisition. The DC-notch filter is used in the spectrally efficient DC-free hierarchical QAM modulation. In optical communication, the DC-notch filter is used to remove the bias and low frequency components from the light receiver. The DC-notch filter is part of the ITU-compliant DTMF detector. The typical application of DC notch filter is in direct conversion receivers to remove the undesired residual offset generated by self-mixing products, and 1/f flicker noise of the down-converted signal.

Comb filters (Chap. 8) are essentially notch filters with deep notches equally spaced in a band of frequencies. The periodic, deep notches make comb filters ideal for applications that need to eliminate specific frequency components, usually interferences. The comb filters find numerous applications in telecommunication systems. In general, telecommunication systems require the elimination of the power-line frequency and its harmonics. In the speech communication, the comb-filter structures are used to determine the echo. In the video communication, the comb filters are used to separate the luminance and chrominance signals and also to reduce video noise. The comb filters were employed in LORAN navigation system for the suppression of cross-rate interferences. Comb filters form the basis of the cascaded integrator-comb (CIC) filters, also known as Hogenauer filters. CIC filters are commonly used as decimation or interpolation filters in digital down and up converters. Some applications that use the CIC filter include software designed radios, cable modems, satellite receivers, 3G base stations and radar systems. The interleaving and complementary comb filters are extensively applied in the interchannel decorrelation methods. Comb filters are essential in powerline communication. However, there are many more applications of the narrow band digital filters in the telecommunication technology not limited to the examples mentioned above.

3. Basic Terms

I will assume the impulse response $h(k)$, $k = 0, \ \dots \ , N-1$, with odd length $N = 2n+1$ and with even symmetry

$$a(0) = h(n) \ \ , \ \ a(k) = 2h(n+k) = 2h(n-k) \, , \ k = 1 \dots n \, . \tag{1}$$

The vector $a(k)$ is more useful for further manipulations than the corresponding impulse response $h(k)$. For brevity I call $a(k)$ the a-vector of the filter. Here and in the following I will use the transformed variable w [22]

$$w = \frac{1}{2}(z + z^{-1}) \tag{2}$$

which transforms the z-plane onto a two-leaved w-plane so that the unit circle itself $\mid z \mid = 1$ is mapped into the real interval $-1 \leq w = \cos \omega T \leq 1$ along which both leaves are interconnected. The transfer function $H(z)$ of a FIR filter of the order $N-1$ is

$$\begin{aligned} H(z) &= \sum_{k=0}^{2n} h(k)\, z^{-k} \\ &= z^{-n}\left[h(n) \; + 2\sum_{k=1}^{n} h(n\pm k)\,\frac{1}{2}\left(z^{k}+z^{-k}\right)\right] = z^{-n}\sum_{k=0}^{n} a(k)\, T_k(w) = z^{-n} Q(w) \end{aligned} \tag{3}$$

where $T_k(w)$ is the Chebyshev polynomial of the first kind. The function

$$Q(w) = \sum_{k=0}^{n} a(k)\, T_k(w) \tag{4}$$

represents a polynomial in the variable w which on the unit circle $z = e^{j\omega T}$ reduces to the real valued zero phase transfer function (ZPTF) $Q(w)$ of the real argument

$$w = \cos(\omega T) \ . \tag{5}$$

The ZPTF is formed by the approximating polynomial (AP). The AP has particular form for each type of filter. The frequency response of the filter $H(e^{j\omega T})$ can be expressed by the ZPTF

$$H(e^{j\omega T}) = e^{-jn\omega T} Q(\cos\omega T) = z^{-n}\; Q(w)\Big|_{z=e^{j\omega T}} \ . \tag{6}$$

4. Analytical Design of Maximally Flat Notch FIR Filters

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4.1. Approximation

The AP of the maximally flat (MF) notch FIR filter is the polynomial $A_{p,q}(w)$

$$A_{p,q}(w) = C(1-w)^{p}(1+w)^{q} \ . \tag{7}$$

The AP $A_{p,q}(w)$ fulfils the differential equation

$$(1-w^2)\frac{dA_{p,q}(w)}{dw} \; + [p-q+(p+q)w]\, A_{p,q}(w) = 0 \ . \tag{8}$$

The differential equation (8) is indispensable for the derivation of algorithm for analytical evaluation of the impulse response. The normalization of the AP $A_{p,q}(w)$ results in

$$A_{p,q}(w) = \left[\frac{p+q}{2p}(1-w)\right]^{p}\left[\frac{p+q}{2q}(1+w)\right]^{q} \ . \tag{9}$$

The polynomial $Q(w) = 1 - A_{p,q}(w)$ represents the ZPTF of the MF notch FIR filter. For illustration, the amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] corresponding to the

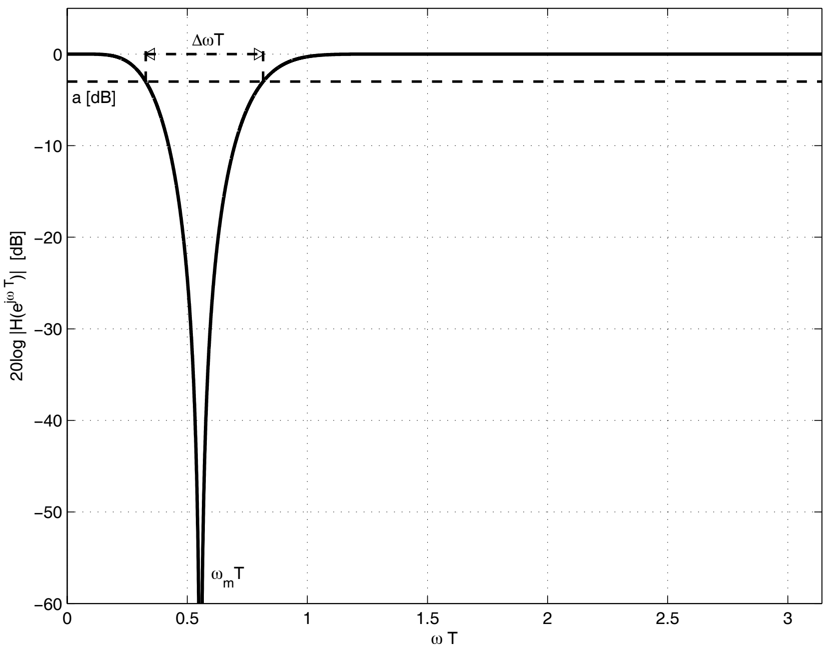

Figure 1: Amplitude frequency response $20\log|H(e^{j\omega T})|$ based on the AP $Q(w) = 1 - A_{3,37}(w)$. The parameters are $\omega_m T = 0.1766\,\pi$ and $\Delta\omega T = 0.1555\,\pi$ for $a = 20\log(\sqrt{2}/2) = -3.0103$ dB.

ZPTF $Q(w) = 1 - A_{3,37}(w)$ is shown in Fig. 1. The notch frequency $\omega_m T$ of the filter is derived from the minimum value of the ZPTF $Q(w)$

$$w_m = \cos\,\omega_m T = \frac{q-p}{q+p} \quad . \tag{10}$$

The notch frequency $\omega_m T$ (10) of the filter is given by the integer values p and q exclusively. It is obvious, that for the specified filter length $N = 2(p+q)+1$, exactly $p+q-1$ discrete notch frequencies $\omega_m T$ are available. The goal in the design of MF notch FIR filter is to obtain the two integers p and q in order to satisfy the filter specification (notch frequency $\omega_m T$, width of the notch band $\Delta\omega T$ and attenuation in the passbands a [dB]) as precisely as possible. The width of the notchband is

$$\Delta\omega T = \pi - 2\arccos\sqrt{1 - \left(1 - 10^{0.05a[dB]}\right)^{2/n}} \tag{11}$$

and the degree of the filter is

$$n \geq \frac{\log\left(1 - 10^{0.05a[dB]}\right)}{\log\cos\dfrac{\Delta\omega T}{2}} \quad . \tag{12}$$

I call (12) the degree equation of the MF notch FIR filter. The integer values p and q are

$$p = \left[n\sin^2\left(\frac{\omega_m T}{2}\right)\right] \quad , \quad q = \left[n\cos^2\left(\frac{\omega_m T}{2}\right)\right] \tag{13}$$

Table 1: Analytical Evaluation of the Impulse Response.

given	$p\,,\;\; q$ (integer values)
initialization	$n = p + q$
	$a(n+1) = 0$
	$a(n) = (-1)^p\; 2^{(-p-q+1)} \left(\dfrac{p+q}{2p}\right)^p \left(\dfrac{p+q}{2q}\right)^q$
recursive body	
(*for* $k = n+1$ *to* 3)	$a(k-2) \;\; = -\dfrac{(n+k)a(k) + 2(2p-n)a(k-1)}{n+2-k}$
(*end loop on* k)	
	$a(0) = -\dfrac{(n+2)a(2) + 2(2p-n)a(1)}{2n}$
impulse response	$h(n) = 1 - a(0)$
(*for* $k = 1$ *to* n)	$h(n \pm k) \;\; = -a(k)/2$
(*end loop on* k)	

where the square brackets stand for the rounding. The AP $A_{p,q}(w)$ of the degree $n = p+q$ can be expressed using Chebyshev polynomials of the first kind $T_k(w)$

$$A_{p,q}(w) = \sum_{k=0}^{n} a(k)\, T_k(w)\ . \tag{14}$$

Based on the differential equation (8) a simple recursive algorithm for the evaluation of the impulse response $h(k)$ with the length $N = 2(p+q)+1$ was deduced (Tab. 1).

4.2. Design Procedure

The design procedure is as follows :

1. Specify the notch frequency $\omega_m T$, maximal width of the notchband $\Delta\omega T$ and maximal attenuation in the passbands a [dB] as demonstrated in Fig. 1.
2. Calculate the minimum degree n (12).
3. Calculate the integer values p and q (13).
4. Evaluate the impulse response $h(k)$ analytically (Tab. 1).
5. Check the notch frequency (10).
6. If required, tune the notch frequency to the proper value (Sec. 6).

It is worth of noting that a substantial part of coefficients of the impulse response $h(k)$ of the MF notch FIR filter exhibits negligible values. From this fact follows the possible large abbreviation of the impulse response of the MF notch FIR filter by the rectangular

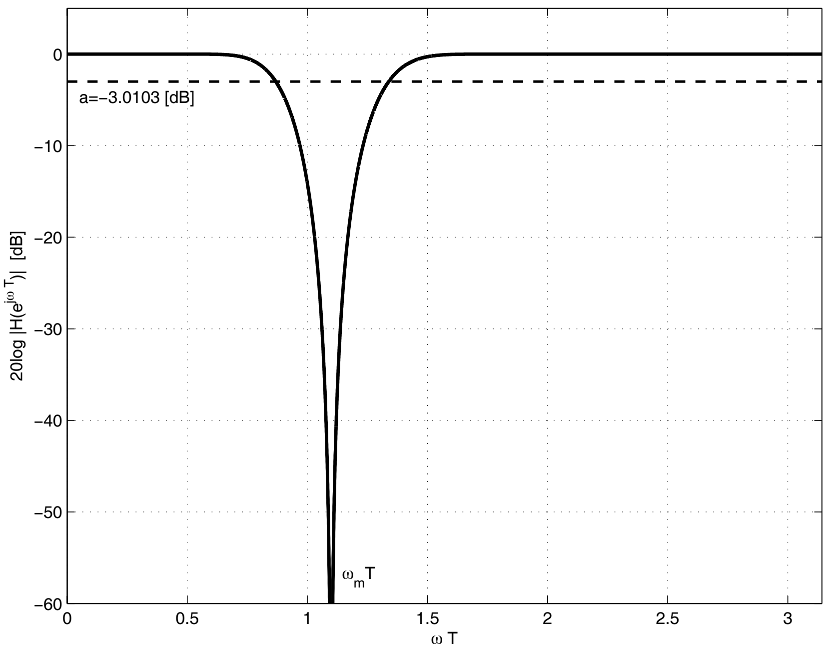

Figure 2: Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

Table 2: Impulse Response $h(k)$.

k		h(k)	k		h(k)
14	74	-0.000002	30	58	0.012289
15	73	-0.000003	31	57	0.002278
16	72	0.000000	32	56	-0.019427
17	71	0.000018	33	55	-0.027483
18	70	0.000037	34	54	-0.003357
19	69	0.000010	35	53	0.042804
20	68	-0.000111	36	52	0.048063
21	67	-0.000245	37	51	-0.009353
22	66	-0.000101	38	50	-0.075616
23	65	0.000537	39	49	-0.065324
24	64	0.001173	40	48	0.029196
25	63	0.000480	41	47	0.106554
26	62	-0.002149	42	46	0.068113
27	61	-0.004302	43	45	-0.053105
28	60	-0.001388	44		0.880514
29	59	0.007135			

windowing without significant deterioration of the frequency properties of the filter as emphasized in [25].

Example of the design.
Design the MF notch FIR filter specified by $\omega_m T = 0.35\,\pi$ *and* $\Delta\omega T = 0.15\,\pi$ *for* $a = -3.0103$ *dB.*

Using the proposed design procedure we get $n = [43.8256] \to 44$ (12), $p = [11.9644] \to 12$ and $q = [31.8610] \to 32$ (13). The actual filter parameters are $\omega_m T = 0.3498\,\pi$ and $\Delta\omega T = 0.1496\,\pi$. The impulse response $h(k)$ with the length $N = 89$ (Tab. 2) was evaluated analytically (Tab. 1). The amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] of the MF notch FIR filter is shown in Fig. 2.

5. Analytical Design of Equiripple Notch FIR Filters

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5.1. Approximation

The GP of the equiripple (ER) notch FIR filter is the Zolotarev polynomial [24]

$$Z_{p,q}(u|k) = \frac{(-1)^p}{2}\left[\left(\frac{H(u-\frac{p}{n}\,\mathbf{K}(\kappa))}{H(u+\frac{p}{n}\,\mathbf{K}(\kappa))}\right)^n + \left(\frac{H(u+\frac{p}{n}\,\mathbf{K}(\kappa))}{H(u-\frac{p}{n}\,\mathbf{K}(\kappa))}\right)^n\right] \tag{15}$$

which approximates zero value in two disjoint intervals. $H\left(u \pm (p/n)\,\mathbf{K}(\kappa)\right)$ is the Jacobi's Eta function, $\mathbf{K}(\kappa)$ is the quarter-period given by the complete elliptic integral of the first kind of the Jacobi's elliptic modulus κ. The degree of the Zolotarev polynomial is $n = p+q$. The indices p and q emphasize that p counts the number of zeros right from the maximum w_m and q corresponds to the number of zeros left from the maximum w_m. The extremal values of the Zolotarev polynomial alternate between -1 and +1 $(q+1)$-times in the interval $(-1, w_s)$ and $(p+1)$-times in the interval $(w_p, 1)$. Assuming the conformal transformation [2], [8] between the u and the w domain

$$w = \frac{\mathrm{sn}^2(u)\mathrm{cn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) + \mathrm{cn}^2(u)\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}{\mathrm{sn}^2(u) - \mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)} \tag{16}$$

I denote $Z_{p,q}(w) = Z_{p,q}(u|\kappa)$ the Zolotarev polynomial in the w-domain. The Zolotarev polynomial $Z_{p,q}(w)$ satisfies the linear differential equation

$$(w-w_p)(w-w_s)(w-w_m)\left[(1-w^2)\frac{d^2Z_{p,q}(w)}{dw^2} - w\frac{dZ_{p,q}(w)}{dw}\right] - \tag{17}$$

$$\left[(w-w_p)(w-w_s) - (w-w_m)(w-\frac{w_p+w_s}{2})\right](1-w^2)\frac{dZ_{p,q}(w)}{dw} + n^2(w-w_m)^3 Z_{p,q}(w) = 0\,.$$

The band edges w_p and w_s correspond to

$$w_p = 2\,\mathrm{sn}^2\left(\frac{q}{n}\mathbf{K}(\kappa)|\kappa\right) - 1\ ,\ \ w_s = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)\ . \tag{18}$$

The position of the maximum value $y_m = Z_{p,q}(w_m)$ is

$$w_m = w_s + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}\mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)\ . \tag{19}$$

The integer values p, q, $n = p + q$ and the real valued elliptic modulus κ are related by the partition equation

$$\frac{p}{n}\mathbf{K}(\kappa) + \frac{q}{n}\mathbf{K}(\kappa) = F(\varphi_s|\kappa) + F(\varphi_p|\kappa) = \mathbf{K}(\kappa)\ . \tag{20}$$

Function $F(\phi|\kappa)$ is the incomplete elliptic integral of the first kind of Jacobi's elliptic modulus κ. The goal in the approximation of the ER notch FIR filter is to obtain the three parameters p, q and κ in order to satisfy the specified notch frequency $\omega_m T$, width of the notchband $\Delta\omega T$ and the attenuation in the passbands a [dB] (Fig. 3) as precisely as possible. The degree of the Zolotarev polynomial is expressed by the degree equation

$$n \geq \frac{\ln(y_m + \sqrt{y_m^2 - 1})}{2\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - 2\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)} \tag{21}$$

where the auxiliary parameter σ_m is

$$\sigma_m = F\left(\arcsin\left(\frac{1}{\kappa\ \mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right)}\sqrt{\frac{w_m - w_s}{w_m + 1}}\right)|\kappa\right)\quad . \tag{22}$$

The maximum value (maximizer) y_m of the Zolotarev polynomial

$$y_m = \cosh 2n\left(\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - \Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)\right) \tag{23}$$

is related to the attenuation in the passbands a [dB]

$$a\,[dB] = 20\log\left(1 - \frac{2}{y_m + 1}\right)\quad . \tag{24}$$

The ZPTF of the ER notch FIR filter is

$$Q(w) = 1 - \frac{Z_{p,q}(w) + 1}{y_m + 1}\quad . \tag{25}$$

Based on the differential equation (17) the algorithm for the evaluation of the a-vector of the Zolotarev polynomial

$$Z_{p,q}(w) = \sum_{k=0}^{n} a(k)T_k(w) \tag{26}$$

and of the impulse response $h(k)$ was developed (Tab. 3).

5.2. Design Procedure

The design procedure is as follows :

1. Specify the notch frequency $\omega_m T$, width of the notchband $\Delta\omega T$ and the maximal attenuation in the passband a [dB] as demonstrated in Fig. 3.
2. Calculate the band edges

$$\omega_p T = \omega_m T - \frac{\Delta\omega T}{2}\ \ ,\ \ \omega_s T = \omega_m T + \frac{\Delta\omega T}{2}\quad . \tag{27}$$

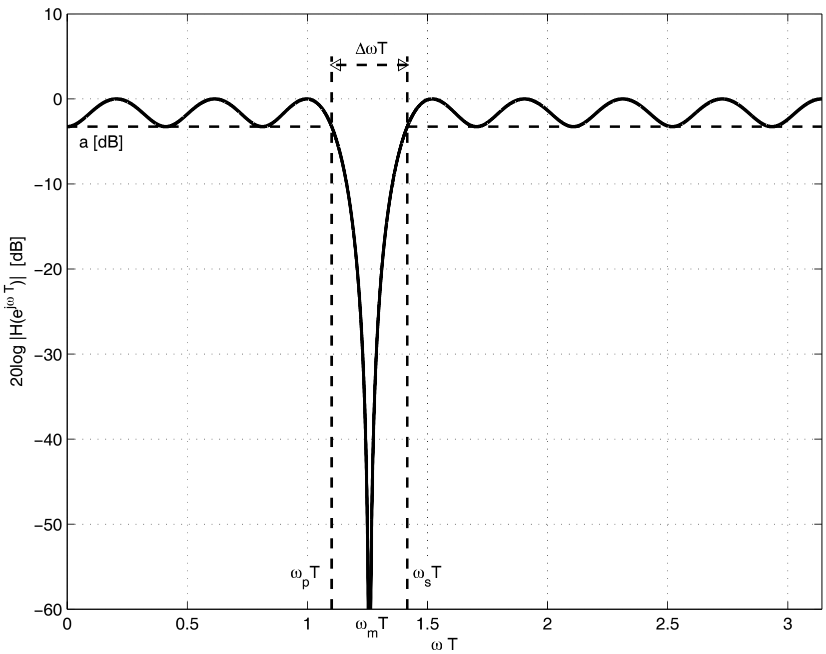

Figure 3: Amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] based on the Zolotarev polynomial $Z_{6,9}(w)$. The parameters are $\omega_p T = 0.3506\,\pi$, $\omega_m T = 0.4006\,\pi$, $\omega_s T = 0.4507\,\pi$, $\Delta\omega T = 0.1001\,\pi$ and $a = -3.2634$ dB.

3. Evaluate the Jacobi's elliptic modulus κ

$$\kappa = \sqrt{1 - \frac{1}{\tan^2(\varphi_s)\tan^2(\varphi_p)}} \tag{28}$$

for the auxiliary parameters φ_s and φ_p

$$\varphi_s = \frac{\omega_s T}{2} \quad , \quad \varphi_p = \frac{\pi - \omega_p T}{2} \quad . \tag{29}$$

4. Calculate the rational values p/n and q/n (20).
5. Determine the required maximum value y_m (23).
6. Calculate the minimum degree n using the degree equation (21).
7. Calculate the integer values p and q defining the Zolotarev polynomial $Z_{p,q}(w)$

$$p = \left[n \frac{F(\varphi_s|\kappa)}{\mathbf{K}(\kappa)} \right] \quad , \quad q = \left[n \frac{F(\varphi_p|\kappa)}{\mathbf{K}(\kappa)} \right] \tag{30}$$

where the square brackets stand for the rounding.
8. Calculate the actual attenuation in the passbands a [dB] (24) for the corresponding maximal value y_m (23).
9. Calculate the actual width of the passband

$$\Delta\omega T = \arccos(w_p) - \arccos(w_s) \quad . \tag{31}$$

10. For p, q and κ evaluate the impulse response $h(k)$ analytically (Tab. 3).

Table 3: Analytical Evaluation of the Impulse Response.

given	$p\,,\, q\,$(integers), $\kappa\,$(real)
initialisation	$n = p + q$
	$w_p = 2\,\mathrm{sn}^2\left(\frac{q}{n}\mathbf{K}(\kappa)\vert\kappa\right) - 1\ ,\ w_s = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)\vert\kappa\right)\ ,\ w_a = \frac{w_p + w_s}{2}$
	$w_m = w_s + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa)\vert\kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa)\vert\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa)\vert\kappa\right)}\mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa)\vert\kappa\right)$
	$\alpha(n) = 1\ ,\ \alpha(n+1) = \alpha(n+2) = \alpha(n+3) = \alpha(n+4) = \alpha(n+5) = 0$
body	
(*for*	$m = n+2$ *to* *3*)
	$8c(1) = n^2 - (m+3)^2$
	$4c(2) = (2m+5)(m+2)(w_m - w_a) + 3w_m[n^2 - (m+2)^2]$
	$2c(3) = \frac{3}{4}[n^2 - (m+1)^2] + 3w_m[n^2 w_m - (m+1)^2 w_a] - (m+1)(m+2)(w_p w_s - w_m w_a)$
	$c(4) = \frac{3}{2}(n^2 - m^2) + m^2(w_m - w_a) + w_m(n^2 w_m^2 - m^2 w_p w_s)$
	$2c(5) = \frac{3}{4}[n^2 - (m-1)^2] + 3w_m[n^2 w_m - (m-1)^2 w_a] - (m-1)(m-2)(w_p w_s - w_m w_a)$
	$4c(6) = (2m-5)(m-2)(w_m - w_a) + 3w_m[n^2 - (m-2)^2]$
	$8c(7) = n^2 - (m-3)^2$
	$\alpha(m-3) = \frac{1}{c(7)}\sum_{\mu=1}^{6} c(\mu)\alpha(m+4-\mu)$
(*end*	*loop on* m)
normalisation	
	$s(n) = \frac{\alpha(0)}{2} + \sum_{m=1}^{n} \alpha(m)$
	$a(0) = (-1)^p \frac{\alpha(0)}{2s(n)}$
(*for*	$m = 1$ *to* n)
	$a(m) = (-1)^p \frac{\alpha(m)}{s(n)}$
(*end*	*loop on* m)
impulse response	$h(n) = \frac{y_m - a(0)}{y_m + 1}$
(*for*	$k = 1$ *to* n)
	$h(n \pm k) = -\frac{a(k)}{2(y_m + 1)}$
(*end*	*loop on* k)

Table 4: Impulse Response $h(k)$.

k		h(k)	k		h(k)
0	76	-0.029617	20	56	0.023583
1	75	0.009026	21	55	-0.016098
2	74	-0.006910	22	54	0.004079
3	73	0.002540	23	53	0.009627
4	72	0.003215	24	52	-0.021585
5	71	-0.008960	25	51	0.028644
6	70	0.013104	26	50	-0.028782
7	69	-0.014293	27	49	0.021678
8	68	0.011819	28	48	-0.008872
9	67	-0.005907	29	47	-0.006575
10	66	-0.002247	30	46	0.020828
11	65	0.010702	31	45	-0.030245
12	64	-0.017226	32	44	0.032332
13	63	0.019891	33	43	-0.026411
14	62	-0.017624	34	42	0.013833
15	61	0.010583	35	41	0.002336
16	60	-0.000216	36	40	-0.018080
17	59	-0.011026	37	39	0.029453
18	58	0.020271	38		0.916626
19	57	-0.024966			

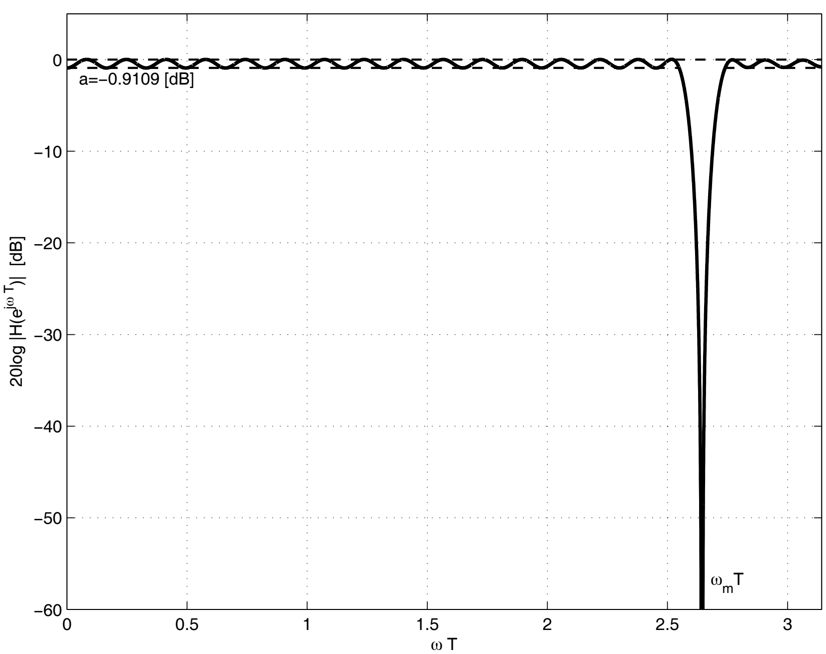

Figure 4: Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

11. Check the notch frequency $\omega_m T = \arccos(w_m)$ using (19).
12. If required, tune the notch frequency to the proper value (Sec. 6).

Example of the design.
Design the ER notch FIR filter specified by the notch frequency $\omega_m T = 0.84\,\pi$ *and by the width of the stopband* $\Delta\omega T = 0.0610\,\pi$ *for the maximal attenuation in the passband*

$a = -0.95\ dB$.
Using the proposed design procedure we get $\omega_p = 0.8095\,\pi$, $\omega_s = 0.8705\,\pi$ (27), $\varphi_s = 0.2992$, $\varphi_p = 1.3674$ (29), $\kappa = 0.743599$ (28), $n = [37.2896] \rightarrow 38$ (21), $p = [31.9713] \rightarrow 32$ and $q = [6.0287] \rightarrow 6$ (30). For the calculated values p, q, κ the actual filter parameters are $\omega_m T = 0.8408\,\pi$ (19), $\Delta\omega T = 0.0607\,\pi$ (31) and $a = -0.9109$ dB (24). The filter length is $N = 77$ coefficients. The impulse response $h(k)$ of the filter evaluated analytically (Tab. 3) is summarized in Tab. 4. The amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] of the filter is shown in Fig. 4.

6. Analytical Procedure for Tuning of FIR Filters

Published in : P. Zahradnik, M. Vlček, "An Analytical Procedure for Critical Frequency Tuning of FIR Filters". *IEEE Transactions on Circuits and Systems II.* January 2006, Vol. 53, No. 1, pp. 72-76 ([30]).

Precise tuning of frequency properties is an useful operation in the design of digital filters. Instead of designing the filter from scratch, the impulse response of the available filter can be reused. Adaptive filtering is one of the applications. Tuning is also useful in the analytical design of digital FIR filters where the available critical frequencies are quantized (Sec. 4 and 5). Hence analytical design combined with tuning the filter represents a powerful design tool. The proposed fast versatile tuning procedure adjusts a single frequency of the frequency response of the FIR filter to the specified value while preserving the nature of the filter, e.g. maximally flat, equiripple etc. The tuning procedure is based on expansion of the Chebyshev polynomial of the transformed argument into the sum of Chebyshev polynomials, resulting in the transformation matrix. The impulse response of the final filter is obtained from the impulse response of the original filter by applying of the transformation matrix. The purpose of the tuning is to map the critical frequency $\omega_m T$ of the frequency response of the filter to the desired value $\omega_0 T$. The mapping $\omega_m T \leftrightarrow \omega_0 T$ in the frequency domain is equivalent to the mapping $w_m \leftrightarrow w_0$ in the w-domain. Due to (5) the shift in the two domains occurs in opposite directions. I propose the transformed ZPTFs in the form

$$Q_t(w) = Q(\lambda w + \lambda') \ , \ \ \lambda = \frac{w_m - 1}{w_0 - 1} \ , \ \text{if } \ \omega_m T < \omega_0 T \tag{32}$$

and

$$Q_t(w) = Q(\lambda w - \lambda') \ , \ \ \lambda = \frac{w_m + 1}{w_0 + 1} \ , \ \text{if } \ \omega_0 T < \omega_m T. \tag{33}$$

The real number λ is confined to $0 < \lambda \leq 1$. The tuning procedure provides the impulse response coefficients for a FIR filter with following properties :

- the frequency $\omega_m T$ is adjusted to the specified value $\omega_0 T$
- the maximal attenuation(s) in the passband(s) and the minimal attenuation(s) of the stopband(s) of the filter are preserved
- the nature (MF, ER etc.) of the filter is preserved and
- the bands of the filter are broadened.

The transformed ZPTFs

$$Q_t(w) = \sum_{k=0}^{n} a(k)\, T_k(\lambda w \pm \lambda^{'}) = \sum_{k=0}^{n} a(k) \sum_{m=0}^{k} \alpha_k(m) T_m(w) \tag{34}$$

can be rewritten in matrix form

$$Q_t(w) = [a(0) \;\; a(1) \;\; \cdots \;\; a(n)] \times \tag{35}$$

$$\begin{bmatrix} \alpha_0(0) & 0 & 0 & 0 & \cdots & 0 \\ \alpha_1(0) & \alpha_1(1) & 0 & 0 & \cdots & 0 \\ \alpha_2(0) & \alpha_2(1) & \alpha_2(2) & 0 & \cdots & 0 \\ \alpha_3(0) & \alpha_3(1) & \alpha_3(2) & \alpha_3(3) & \cdots & 0 \\ \vdots & & & & & \vdots \\ \alpha_n(0) & \alpha_n(1) & \alpha_n(2) & \alpha_n(3) & \cdots & \alpha_n(n) \end{bmatrix} \times \begin{bmatrix} T_0(w) \\ T_1(w) \\ T_2(w) \\ T_3(w) \\ \vdots \\ T_n(w) \end{bmatrix} \;=\; a \;\; A \;\; \mathcal{T} \;\; . \tag{36}$$

I call the low triangular matrix A the transformation matrix. The a_t-vector of the transformed filter is given by the product of a-vector of the original filter and the transformation matrix A

$$a_t = a \;\; A \;\; . \tag{37}$$

There are two transformation matrices A_+ and A_- corresponding to the transformed ZPTFs (32) and (33). Fast evaluation of the coefficients $\alpha_k(m)$ of the transformation matrices is essential in adaptive filtering. Based on the differential equation for the Chebyshev

Table 5: Evaluation of the Coefficients $\alpha_k(m)$ of the Transformation Matrix A_+.

given	k (integer value), $0 < \lambda \leq 1$ (real value)
initialization	$\lambda^{'} = 1 - \lambda$ $\alpha_k(k+1) = \alpha_k(k+2) = \alpha_k(k+3) = 0$ $\alpha_k(k) = \lambda^k$
body	
($for\ \mu = -3\ \ldots\ k-4$)	
	$\alpha_k(k-\mu-4) =$ $\{$ $-2\left[(\mu+3)(2k-\mu-3) - \frac{\lambda^{'}}{\lambda}(k-\mu-3)(2k-2\mu-7)\right] \alpha_k(k-\mu-3)$ $+2\frac{\lambda^{'}}{\lambda}2(k-\mu-2)\ \alpha_k(k-\mu-2)$ $+2\left[(\mu+1)(2k-\mu-1) - \frac{\lambda^{'}}{\lambda}(k-\mu-1)(2k-2\mu-1)\right] \alpha_k(k-\mu-1)$ $+\mu(2k-\mu)\ \alpha_k(k-\mu)$ $\}\ /\ (\mu+4)(2k-\mu-4)$
($end\ loop\ on\ \mu$)	

polynomial of the first kind $T_k(x)$

$$(1-x^2)\frac{d^2T_k(x)}{dx^2} - x\frac{dT_k(x)}{dx} + k^2T_k(x) = 0 \tag{38}$$

Table 6: Impulse Responses $h(k)$ and $h_t(k)$.

k		h(k)	h_t(k)	k		h(k)	h_t(k)
0	72	0.016832	0.011622	19	53	0.022942	0.028084
1	71	0.004953	-0.005198	20	52	0.028636	0.024800
2	70	-0.002260	-0.009660	21	51	0.009196	0.000144
3	69	-0.009076	-0.010249	22	50	-0.019871	-0.026360
4	68	-0.008700	-0.003681	23	49	-0.033269	-0.032071
5	67	0.000117	0.006768	24	48	-0.018035	-0.010712
6	66	0.010632	0.013012	25	47	0.014008	0.021071
7	65	0.013100	0.008921	26	46	0.035383	0.036681
8	64	0.003678	-0.003396	27	45	0.026667	0.021933
9	63	-0.010795	-0.013938	28	44	-0.005787	-0.012097
10	62	-0.017533	-0.012654	29	43	-0.034396	-0.037428
11	61	-0.009034	0.001287	30	42	-0.033926	-0.032360
12	60	0.009012	0.017271	31	41	-0.003929	-0.000239
13	59	0.021195	0.021210	32	40	0.030153	0.032628
14	58	0.015517	0.007859	33	39	0.038727	0.038693
15	57	-0.004980	-0.013249	34	38	0.013946	0.012424
16	56	-0.023239	-0.024485	35	37	-0.023124	-0.024678
17	55	-0.022375	-0.014929	36		0.933816	0.932507
18	54	-0.001244	0.009158				

I have derived the differential equation

$$(1 - w^2 + 2\frac{\lambda^{'}}{\lambda}(1-w))\frac{d^2F_+(w)}{dw^2} - (w + \frac{\lambda^{'}}{\lambda})\frac{dF_+(w)}{dw} + k^2F_+(w) = 0 \tag{39}$$

for the polynomial

$$F_+(w) = T_k(\lambda w + \lambda^{'}) \tag{40}$$

and the differential equation

$$(1 - w^2 + 2\frac{\lambda^{'}}{\lambda}(1+w))\frac{d^2F_-(w)}{dw^2} - (w - \frac{\lambda^{'}}{\lambda})\frac{dF_-(w)}{dw} + k^2F_-(w) = 0 \tag{41}$$

for the polynomial

$$F_-(w) = T_k(\lambda w - \lambda^{'}) \tag{42}$$

where the real values λ and $\lambda^{'}$ are related by $\lambda + \lambda^{'} = 1$. Based on differential equations (39) and (41) I have derived a fast procedure for evaluation of the coefficients $\alpha_k(m)$ of the transformation matrices A_+ and A_-. The fast algorithm for the evaluation of the coefficients of the transformation matrix A_+ is summarized in Tab. 5. The evaluation of the transformation matrix A_- is by analogy. Both matrices differ by the signs of the "odd" coefficients $\alpha_k(k-\mu-3)$ and $\alpha_k(k-\mu-1)$ only.

Example of the tuning.
Design the ER notch FIR filter specified by the notch frequency $\omega_0 T = 0.3\,\pi$ and width of the notchband $\Delta\omega T = 0.075\,\pi$ for maximal attenuation in the passbands $a = -0.5$ dB.
Using the analytical design procedure (Sec. 5) we get $\kappa = 0.665619$, $n = 36$, $p = 11$ and $q = 25$. The designed filter of length $N = 73$ coefficients with "quantized" notch

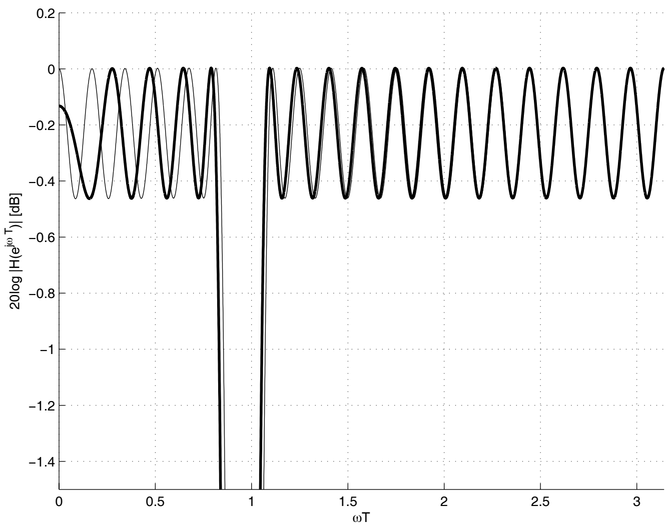

Figure 5: Passbands of the "quantized" (thin line) and of the tuned filter.

frequency $\omega_m T = 0.3064\,\pi$ and $\Delta\omega T = 0.075\,\pi$ for $a_{act} = -0.4584$ dB will be tuned using the proposed tuning procedure in order to get the specified notch frequency $\omega_0 T = 0.3\,\pi$. Because $\omega_0 T < \omega_m T$ we evaluate the transformation matrix A_- for $\lambda = 0.9898$ (33). The parameters of the tuned filter are $\omega_0 T = 0.3\,\pi$ and $\Delta\omega T = 0.0779\,\pi$ for $a = -0.4584$ dB. A detailed view of the passbands of the "quantized" and of the tuned filter is shown in Fig. 5.

7. Analytical Design of Equiripple DC-Notch FIR Filters

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7.1. Approximation

The approximating polynomial of the DC-notch FIR filter is the polynomial $F(w)$

$$F(w) = T_n(\lambda\, w + \lambda - 1) = \sum_{m=0}^{n} B(m)\, w^m = \sum_{m=0}^{n} A(m)\, T_m(w) \quad . \tag{43}$$

The approximating polynomial $F(w)$ fulfils the differential equation

$$(1 - w^2 + 2\frac{1-\lambda}{\lambda})\frac{d^2F(w)}{dw^2} - (w - \frac{1-\lambda}{\lambda})\frac{dF(w)}{dw} + n^2\, F(w) = 0 \quad . \tag{44}$$

The differential equation (44) is indispensable for the derivation of algorithm for analytical evaluation of the impulse response of the filter. The zero phase transfer function of the

DC-notch FIR filter is

$$Q(w) = 1 - \frac{F(w)+1}{T_n(2\lambda-1)+1} = 1 - \frac{T_n(\lambda\, w + \lambda - 1)+1}{T_n(2\lambda-1)+1} \quad . \tag{45}$$

The DC-notch FIR filter is specified by the passband frequency $\omega_p T$ and by the attenuation in the passband a [dB] (Fig. 6). The degree equation reads as follows

$$n \geq \frac{\operatorname{acosh} \dfrac{1+10^{0.05a[dB]}}{1-10^{0.05a[dB]}}}{\operatorname{acosh} \dfrac{1+\sin^2 \dfrac{\omega_p T}{2}}{1-\sin^2 \dfrac{\omega_p T}{2}}} \tag{46}$$

where

$$\lambda = \frac{1}{1-\sin^2 \dfrac{\omega_p T}{2}} \quad . \tag{47}$$

Based on the differential equation (44) the algorithm for the evaluation of the impulse response of the DC-notch FIR filter was derived. The algorithm is summarized in Tab. 7.

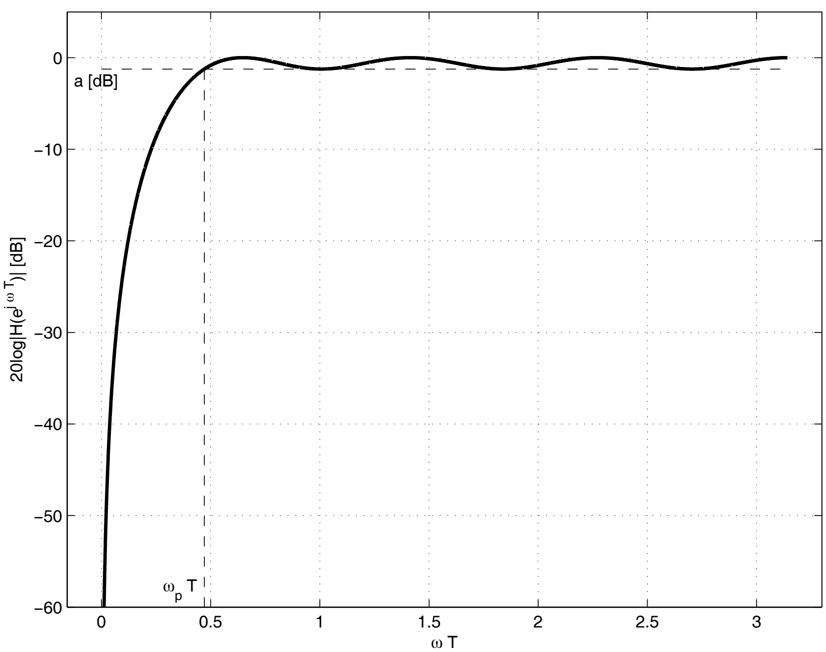

Figure 6: Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] for $n = 7$, $\lambda = 1.057638$, $\omega_p T = 0.15\pi$ and $a = -1.2446$ dB.

Table 7: Analytical Evaluation of the Impulse Response.

given	n (integer value), λ (real value)
initialization	$A(n) = \lambda^n$, $A(n+1) = A(n+2) = A(n+3) = 0$
body ($for\ k = 2\ \dots\ n+1$)	$A(n+1-k) =$
	$\{\ 2\,[(k-1)(2n+1-k) - ((1-\lambda)/\lambda)(n+1-k)(2n+1-2k)]\,A(n+2-k)$ $+\,4\,((1-\lambda)/\lambda)(n+2-k)\ \ A(n+3-k)$ $-\,2\,[(k-3)(2n+3-k) - ((1-\lambda)/\lambda)(n+3-k)(2n+7-2k)]\ A(n+4-k)$ $+\,(k-4)(2n+4-k)\ \ A(n+5-k)\quad \}\ /\ k(2n-k)$
($end\ loop\ on\ k$)	$h(0) = 1 - \dfrac{A(0)/2+1}{T_n(2\lambda-1)+1}$
	$h(\pm k) = -\dfrac{1}{2}\dfrac{A(k)+1}{T_n(2\lambda-1)+1}$, $k = 1 \dots n$

Table 8: Impulse Response $h(k)$.

k	h(k)	k	h(k)
0 , 104	-0.000387	27 , 77	-0.009226
1 , 103	-0.000248	28 , 76	-0.009866
2 , 102	-0.000325	29 , 75	-0.010516
3 , 101	-0.000416	30 , 74	-0.011173
4 , 100	-0.000523	31 , 73	-0.011834
5 , 99	-0.000646	32 , 72	-0.012495
6 , 98	-0.000787	33 , 71	-0.013154
7 , 97	-0.000947	34 , 70	-0.013807
8 , 96	-0.001128	35 , 69	-0.014451
9 , 95	-0.001330	36 , 68	-0.015081
10 , 94	-0.001556	37 , 67	-0.015696
11 , 93	-0.001805	38 , 66	-0.016291
12 , 92	-0.002079	39 , 65	-0.016862
13 , 91	-0.002378	40 , 64	-0.017407
14 , 90	-0.002704	41 , 63	-0.017921
15 , 89	-0.003056	42 , 62	-0.018402
16 , 88	-0.003435	43 , 61	-0.018848
17 , 87	-0.003840	44 , 60	-0.019254
18 , 86	-0.004273	45 , 59	-0.019619
19 , 85	-0.004731	46 , 58	-0.019941
20 , 84	-0.005216	47 , 57	-0.020216
21 , 83	-0.005725	48 , 56	-0.020444
22 , 82	-0.006258	49 , 55	-0.020623
23 , 81	-0.006813	50 , 54	-0.020752
24 , 80	-0.007390	51 , 53	-0.020829
25 , 79	-0.007986	52	0.978583
26 , 78	-0.008598		

7.2. Design Procedure

The design procedure is as follows :

1. Specify the passband frequency $\omega_p T$ and the maximal attenuation in the passband a [dB] as demonstrated in Fig. 6.
2. Calculate the minimum degree n (46).
3. Evaluate the impulse response $h(k)$ analytically (Tab. 7).

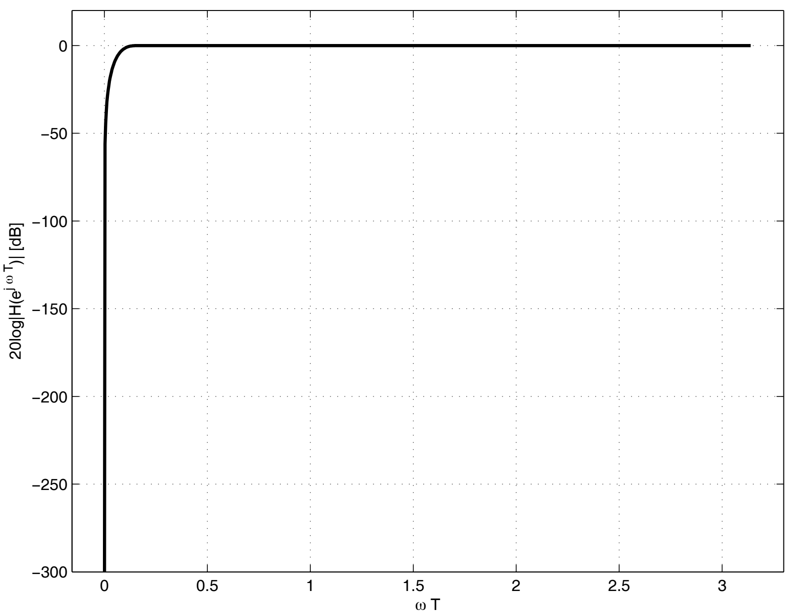

Figure 7: Amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB].

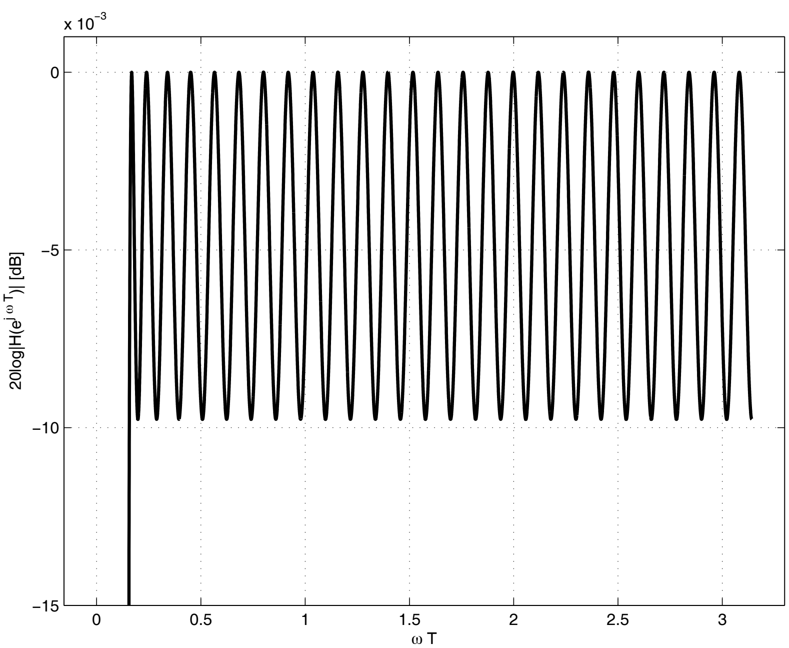

Figure 8: Passband of the filter.

Example of the design.
Design the DC-notch FIR filter specified by $\omega_p T = 0.05\pi$ and $a = -0.01\,dB$.
Using the proposed design procedure we get $n = 51.8513 \rightarrow 52$ (46) and $\lambda = 1.006194$ (47). The actual filter parameters are $\omega_p T = 0.05\pi$ and $a_{act} = -0.009768$ dB. The impulse response $h(k)$ (Tab. 8) with the length $N = 105$ was evaluated analytically (Tab. 7). The amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] of the DC-notch FIR filter is shown in Fig. 7. Its passband is shown in Fig. 8.

8. Analytical Design of Equiripple Comb FIR Filters

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8.1. Approximation

The GP $F(w)$ of the ER comb FIR filter is given by the compounded Chebyshev polynomial

$$F(w) = T_n\left[\lambda\, T_r(w)\right] = \sum_{k=0}^{nr} A(k)\, T_k(w) \quad . \tag{48}$$

The real parameter $\lambda = 1/\kappa > 1$ affects the ripples in the passbands of the comb FIR filter. The degree r of the inner Chebyshev polynomial determines r narrow bands. The narrow bands of the comb FIR filter positioned at

$$\omega_{mi}\, T = \frac{i\,\pi}{r} \quad i = 0, 1, \, \ldots\, , r \tag{49}$$

are equally spaced inside the interval $[0, \pi]$. The even degree n of the outer Chebyshev polynomial $T_n(w)$ determines $n - 1$ local extremes with the same amplitude between the narrow bands. The GP $F(w)$ (48) of the ER comb FIR filter fulfils the differential equation

$$\begin{aligned} U_{r-1}(w)\left(\kappa^2 - T_r^2(w)\right)&\left[(1-w^2)\frac{d^2F(w)}{dw^2} - w\frac{dF(w)}{dw}\right] \\ &-r(1-\kappa^2)T_r(w)\frac{dF(w)}{dw} + n^2r^2U_{r-1}(w)\left(1 - T_r^2(w)\right)F(w) = 0 \end{aligned} \tag{50}$$

where $U_r(w)$ is the Chebyshev polynomial of the second kind. The differential equation (50) is indispensable for the derivation of algorithm for analytical evaluation of the impulse response of the filter. The ZPTF $Q(w)$ of the comb filter is given by the normalization of the GP

$$Q(w) = 1 - \frac{1 + F(w)}{C} = 1 - \frac{1 + T_n\left[\lambda\, T_r(w)\right]}{C} = \sum_{k=0}^{nr} a(k)\, T_k(w)\, . \tag{51}$$

The normalizing constant C follows from the GP $F(w)$ for $w = 1$

$$C = 1 + F(1) \;\; = 1 + T_n\left[\lambda\, T_r(1)\right] = 1 + T_n\left(\lambda\right) = 1 + \cosh[n\, \mathrm{acosh}(\lambda)]\, . \tag{52}$$

Note that (52) is independent from the degree r of the inner Chebyshev polynomial $T_r(w)$. The goal in the approximation of the ER comb FIR filter is to obtain the two parameters

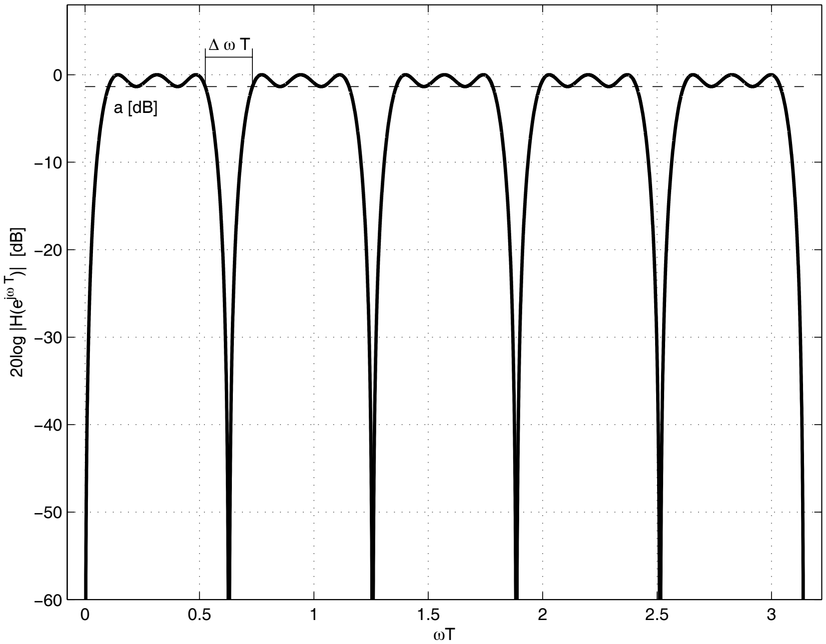

Figure 9: Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

n and λ in order to satisfy the specified number of notch bands r, the width of the notch bands $\Delta\omega T$ and the maximal attenuation in the passbands a [dB] (Fig. 9) as precisely as possible. The degree n of the outer Chebyshev polynomial $T_r(w)$ is

$$n \geq \frac{\operatorname{acosh}(\chi)}{\operatorname{acosh}(\lambda)} = \frac{\ln(\chi + \sqrt{\chi^2 - 1})}{\ln(\lambda + \sqrt{\lambda^2 - 1})} \tag{53}$$

where the parameters λ and χ are

$$\lambda = \frac{1}{\cos\left(r\dfrac{\Delta\omega T}{2}\right)} \quad , \quad \chi = \frac{1 + 10^{0.05\mathrm{a[dB]}}}{1 - 10^{0.05\mathrm{a[dB]}}} \quad . \tag{54}$$

I call (53) the degree equation of the ER comb FIR filter. The real value n (53) has to be up-rounded to the next even integer value. This up-rounding preserves the specified number of notch bands and the width of the notchbands. The attenuation in passbands a [dB] is equal or less than the specified value. The impulse response $h(k)$ of the filter consists of $2nr + 1$ coefficients, among them are $n + 1$ non-zero values. For illustration, the amplitude frequency response $20\log|H(e^{j\omega})|$ [dB] based on the ZPTF

$$Q(w) = 1 - \frac{1 + T_6\left[1.15\, T_5(w)\right]}{1 + T_6\left(1.15\right)} \tag{55}$$

is shown in Fig. 9. Note that there are true zeros at the notch frequencies. Based on the differential equation (50) a simple analytical algorithm for the algebraic evaluation of the impulse response $h(k)$ of the filter was developed (Tab. 9).

8.2. Design Procedure

The design procedure for the ER comb FIR filter consists of the following steps :

1. Specify the number of notch bands r, the width of the notch bands $\Delta\omega T$ and the maximal attenuation in the passbands a [dB] as demonstrated in Fig. 9.
2. The degree of the inner Chebychev polynomial is r.
3. Determine the auxiliary parameters λ and χ (54).
4. Evaluate the real value n (53) and round it to the next even integer value.
5. Evaluate the impulse response $h(k)$ analytically (Tab. 9).

Example of the design.
Design an ER comb FIR filter with 20 notch bands specified by the width of the notch bands $\Delta\omega T = \pi/50$ *and by the maximal attenuation in the passbands* $a = -1$ *dB.*
The degree of the inner Chebyshev polynomial is $r = 20$. We get $\lambda = 1.2361$, $k = 17.3910$ (54) and $n = 5.2623 \rightarrow 6$ (53). The ZPTF is

$$Q(w) = 1 - \frac{1 + T_6\,[1.2361\,T_{20}(w)]}{1 + T_6\,(1.2361)} \quad . \tag{56}$$

The impulse response $h(k)$ with a length of 241 coefficients is evaluated analytically (Tab. 9). It consists of seven non-zero coefficients only. It is summarized in Tab. 10. The actual parameters of the comb FIR filter are $\Delta\omega T = \pi/50$ and $a = -0.6080$ dB. The amplitude frequency response $20\log|H(e^{j\omega})|$ [dB] of the filter is shown in Fig. 10.

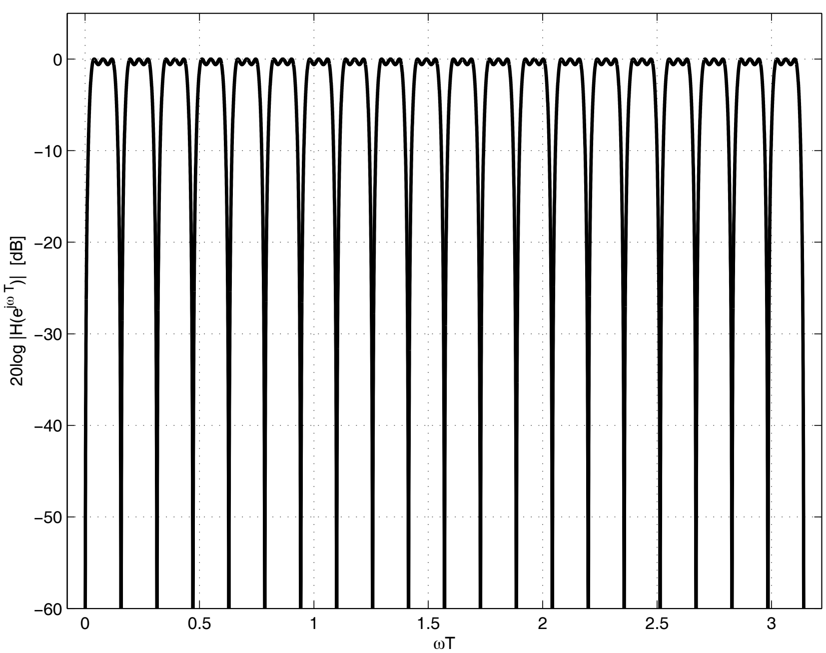

Figure 10: Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

Table 9: Analytical Evaluation of the Impulse Response.

given n (even integer), r (integer), $\lambda > 1$ (real)

initialization $\kappa = \frac{1}{\lambda} \quad , \quad \alpha(n) = \lambda^n \quad , \quad \alpha(n+2) = \alpha(n+4) = \alpha(n+6) = 0$

body

(*for* $k = 1 \ \ldots \ \frac{n}{2}$)

$$\begin{aligned} \alpha(n-2k) = \\ \{ \quad & \alpha(n-2(k-1)) \times \\ & [(1-\kappa^2)(n-(2k-1))(n-(2k-2)) + 3(k-1)(n-(k-1))] \\ & -\alpha(n-2(k-2)) \times \\ & [(1-\kappa^2)(n-(2k-4))(n-(2k-5)) + 3(k-2)(n-(k-2))] \\ & +\alpha(n-2(k-3))(k-3)(n-(k-3)) \quad \} \ / \ k(n-k) \end{aligned}$$

(*end loop on* k)

$$\alpha(0) = \frac{\alpha(0)}{2}$$

$A(k)$ of $F(w)$

body

(*for* $k = 0 \ \ldots \ \frac{n}{2}$)

$$A(nr - 2kr) = \alpha(n - 2k)$$

(*end loop on* k)

$a(k)$ *of* $Q(w)$ $C = \cosh[n \, \mathrm{acosh}(\lambda)] \quad , \quad a(0) = 1 - \frac{1 + A(0)}{C}$

body

(*for* $k = 1 \ \ldots \ nr$)

$$a(k) = -\frac{A(k)}{C}$$

(*end loop on* k)

impulse response $h(k)$

$$h(nr) = a(0)$$

body

(*for* $k = 1 \ \ldots \ nr$)

$$h(nr \pm k) = \frac{a(k)}{2}$$

(*end loop on* k)

Table 10: Non-zero Coefficients of the Impulse Response $h(k)$.

k		h(k)
0	240	-0.060281
40	200	-0.124960
80	160	-0.189719
120		0.749920

9. Robustness of the Analytical Design

It is worth of noting that the analytical design by far outperforms the numerical procedures (e.g. [9]). In order to demonstrate the robustness, we wish to design the equiripple DC-notch FIR filter with quite absurd specification $\omega_p T = 0.00001\pi$ and $a = -0.01$[dB]. Using the proposed analytical procedure (Sec. 7), we get $\lambda = 1.00000000024674$, $n = 259523.24 \rightarrow 259524$. The length of the filter amounts $N = 519049$ coefficients. The zero phase transfer function of the filter is

$$Q(w) = 1 - \frac{T_{259524}\,(1.00000000024674\,w + 0.00000000024674) + 1}{T_{259524}(1.00000000049348) + 1} \ . \tag{57}$$

The properties of the designed filter are $\omega_p T = 0.00001\pi$ and $a = -0.00999976$[dB]. The amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] of the filter is shown in Fig. 11.

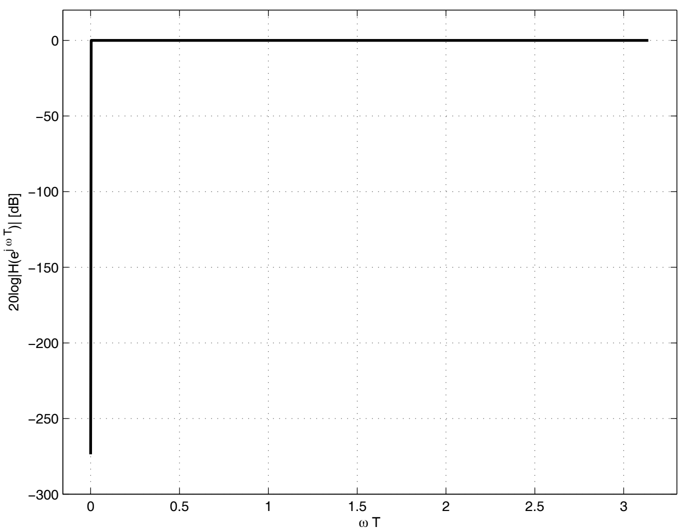

Figure 11: Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

10. Concluding Remarks

In this document I have introduced novel purely analytical solutions for the design of maximally flat notch FIR filters, equiripple notch FIR filters, equiripple DC-notch FIR filters and equiripple comb FIR filters for telecommunication applications. Formulae for the degree and for the impulse response of the filters were presented. The design procedures were published in four IEEE papers. My future work may comprise the development of the analytical design procedures for multiple notch FIR filters with non-equidistantly located notch frequencies, FIR filters with maximally flat pass band(s) and equiripple stop band(s) and non-narrow band FIR filters.

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11. Curriculum Vitae

Name : Doc. Ing. Pavel Zahradník, CSc.
Born : January 17th, 1963, Mělník, ČSSR
Address : Czech Technical University Prague (ČVUT), dept. of Telecommunication Engineering, E-mail : zahradni @ fel.cvut.cz
Education : 1986 - M.Sc. (Ing.) in Microelectronics, ČVUT
1991 - Ph.D. (CSc.) in Telecommunication Engineering, ČVUT
Languages : English, German

Professional carrier
1986-1990 postgraduate student, dept. of Circuit Theory, ČVUT
1991-2001 assistant professor, dept. of Circuit Theory, ČVUT
2001-2004 assistant professor, dept. of Telecommunication Engineering, ČVUT
2004- associated professor (Doc.), dept. of Telecommunication Engineering, ČVUT

External experience
10/1993 - 5/1994 (8 months), Paul Scherrer Institute Villigen and University of Technology Zürich (ETH), research in microwave tomography for localisation of tumors, c/o Hans-Ulrich Boksberger
11/1995 - 12/1995 (2 months), fellowship Goethe Institut München
1/1996 - 12/1996, **1/1997 - 6/1997** and **3/1998 - 8/1998** (24 months), three fellowships Alexander von Humboldt Stiftung, Lehrstuhl für Allgemeine und Theoretische Elektrotechnik, Universität Erlangen-Nürnberg, research in digital filters, c/o Prof. Dr.-Ing. Dr.h.c. Rolf Unbehauen
8/2000 - 9/2000 (2 months), DAAD/NATO fellowship, Lehrstuhl für Allgemeine und Theoretische Elektrotechnik, Universität Erlangen-Nürnberg, research in digital filters, c/o Prof. Dr.-Ing. Dr.h.c. Rolf Unbehauen

Teaching experience
Lectures : Microprocessor technology in telecommunication engineering, Digital signal processing, Implementation of algorithms using digital signal processors
Supervising of graduate students : over 50 successfully graduated students
Supervising of Ph.D. students : 16 PhD students, 2 successful PhD students, currently 5 PhD students

Publications : over 80 journal and conference papers

International grants
1996, 1997, 1998 - 3 individual research grants, Alexander von Humboldt Stiftung
1996 - Grant Texas Instruments, DSP system TMS320C80 for image processing
2000 - Grant Alexander von Humboldt Stiftung, Equipment for DSP in thermography
2004 - Grant Texas Instruments, Equipment for the TI DSP Lab
2005 - Grant Texas Instruments, Design tools for the TI DSP Lab
Research areas : design of digital filters, digital signal and image processing, image restauration, compression techniques, digital signal processors
Miscellaneous : chairmanship on international conferences, reviews for international conferences, reviews for IEEE Transactions, member of IEEE, constituent member of European Circuit Society