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Teorie pravděpodobnosti na kvantových a fuzzy logikách

Probability Theory on Quantum and Fuzzy Logics

Summary

The classical probability theory was successful in many tasks and became important mainly as the basis of statistics. However, there are at least two reasons for its revision. Some systems violate the assumptions of the classical theory and require a more general probability model. This brings new mathematical problems worth attention.

Quantum logic, as a basis of quantum probability, deals with events which can be observed separately but not simultaneously. This implies that it is unnecessary (and even impossible) to define the probability of their conjunction and other formulas which were meaningful in the classical theory. A generalized model (based on lattices of subspaces of Hilbert spaces or, more generally, on orthomodular lattices) allows to explain the paradoxes of quantum mechanics. Nevertheless, it is applicable also in other fields, e.g., sociology, psychology, artificial intelligence, etc.

We study the space of probabilities (=states) on quantum structures and simplify the description of state spaces. This allows new constructions answering theoretical problems. Among others, we give a negative answer to the uniqueness problem for bounded observables (formulated by Gudder in 1966): The model admits two different observables (=quantum random variables) which have the same expectations in all states and thus cannot be distinguished by any measurement. We also prove that the state space is independent of other attributes of the quantum system, in particular its symmetries, subsystems, and the classical part of the system. We refine arguments proving the impossibility of embedding of a quantum system into a classical one.

Fuzzy logic represents another generalization. It deals with events whose truth can be evaluated by more than two values. This approach allows to represent vagueness of information. Imprecise quantities can be determined in terms of fuzzy logic similarly as in natural language. Therefore fuzzy sets are used to translate human reasoning into algorithms applied in control and decision making.

Probability theory on fuzzy sets requires first to describe the underlying structure—a tribe—which is a generalization of the classical notion of σ-algebra of sets. We contribute to this study by description of tribes in the most important cases. Then we characterize measures on tribes. This generalizes previous results on probability theory on fuzzy sets which were successfully applied to games with fuzzy coalitions. The study of fuzzy logical operations enabled also a comparison of semantics of various fuzzy logics. We also contributed to the interpretation of rules in fuzzy controllers and improved some of their applications.

It is shown that various types of uncertainty differ not only in their origin, but necessarily also in the structures representing them.

Souhrn

Klasický pravděpodobnostní model byl úspěšný v mnoha aplikacích, zejména jako základ statistiky. Nicméně existují nejméně dva důvody pro jeho revizi. Některé systémy porušují předpoklady klasické teorie a vyžadují obecnější model. To přináší nové matematické problémy hodné pozornosti.

Kvantová logika jako základ kvantové pravděpodobnosti se zabývá existencí jevů, které lze pozorovat odděleně, nikoli však současně. Z toho vyplývá, že není nutné (ba ani možné) definovat pravděpodobnost jejich konjunkce a dalších formulí, které měly v klasické teorii smysl. Zobecněný model (založený na svazech podprostorů Hilbertových prostorů nebo obecněji na ortomodulárních svazech) dovoluje vysvětlit paradoxy kvantové mechaniky. Je však použitelný i v jiných oblastech, např. v sociologii, psychologii, umělé inteligenci atd.

Zde studujeme prostor pravděpodobností (=stavů) na kvantových strukturách a zjednodušujeme popis stavových prostorů. To dovoluje konstrukce řešící teoretické problémy. Mj. dáváme negativní odpověď na problém jednoznačnosti pozorovatelných (formulovaný Gudderem v r. 1966): existují dvě různé pozorovatelné (=kvantové náhodné veličiny), které mají stejnou střední hodnotu ve všech stavech a tudíž jsou měřením nerozlišitelné. Dále ukazujeme, že stavový prostor nezávisí na dalších atributech, jako jsou symetrie, podsystémy či klasická část systému. Přispíváme k vylepšení argumentů dokazujících nemožnost vnoření kvantového systému do klasického.

Fuzzy logika představuje jiné zobecnění. Zabývá se jevy, jejichž pravdivost může být ohodnocena více než dvěma hodnotami. Tento přístup umožňuje reprezentaci vágních informací. Nepřesné kvantity lze vyjádřit ve fuzzy logice podobně jako v přirozeném jazyce. Proto jsou fuzzy množiny používány k převádění lidských metod uvažování na algoritmy aplikované v řízení a rozhodování.

Teorie pravděpodobnosti na fuzzy množinách vyžaduje především popis základních struktur zobecňujících klasický pojem σ-algebry množin. Přispíváme jejich charakterizací v nejdůležitějších případech. Dále jsou charakterizovány pravděpodobnosti na tomto modelu. Tím jsou zobecněny předchozí výsledky úspěšně použité v teorii her s fuzzy koalicemi. Studium fuzzy logických operací umožnilo také srovnání sémantiky různých fuzzy logik. Práce přispívá též k interpretaci pravidel ve fuzzy regulátorech a vylepšení některých jejich aplikací.

Je ukázáno, že různé typy neurčitosti se liší nejen svým původem, ale nutně i strukturami, které je reprezentují.

Klíčová slova: pravděpodobnostní míra, stav, kvantová logika, ortomodulární svaz, hilbertovský svaz, stavový prostor, pozorovatelná, symetrie, skryté proměnné, fuzzy logika, fuzzy řízení, typy neurčitosti.

Keywords: probability measure, state, quantum logic, orthomodular lattice, Hilbert lattice, state space, observable, symmetry, hidden variables, fuzzy logic, tribe, fuzzy control, types of uncertainty.

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1 Motivation

The classical probability model due to Kolmogorov has been used in many fields, in particular as a basis of mathematical statistics. Beside numerous successful applications, difficulties occurred during attempts to describe some systems. The reason is that these systems violate some of the axioms of the classical probability theory.

Quantum mechanics has been the first field which required a revision of the probability theory. Due to the uncertainty principle, some events cannot be tested simultaneously. Therefore there is no reason to assign a probability to their conjunction (disjunction, etc.) if such a phenomenon is not observable. This gives us more freedom in the probabilistic description of the system. Without this modification, the theory did not allow to explain phenomena occurring in quantum physics.

Quantum probability required a rather different description developed mainly by Heisenberg and Schrödinger. Von Neumann [91] has found a mathematical formalism describing both previous approaches. It is based on a Hilbert space. Although this attempt was very successful, even its author was not fully satisfied with it. In co-operation with Birkhoff [8], he tried to find algebraic conditions leading to this Hilbert space formalism as a unique solution of well-formulated and motivated conditions on a quantum system. With this intention, they introduced orthomodular lattices as a common generalization of the classical and quantum probability model. This effort led to important mathematical results which, however, did not fulfil its original purpose completely. This line of research has been paid much attention after the publication of Mackey's book [63]. Numerous experts have found systems of conditions which select the Hilbert space or von Neumann algebra models among general orthomodular lattices [10, 11, 93, 99]. Despite great attention, none of these systems is completely satisfactory; each of them contains some artificial conditions whose motivation and interpretation is questionable. Nevertheless, this effort has brought many important results, in particular the highly advanced Gleason's theorem [30]. Now probability theory on quantum logics (so-called *non-commutative measure theory*) and the theory of the corresponding algebraic structures—orthomodular lattices and their generalizations—is a well-established field of mathematics which is subject to rapid development [6, 7, 22, 23, 37, 39, 52, 97, 112, 113, 102].

Another generalization of the classical probability theory tries to describe vagueness of data. There are experiments whose results cannot be satisfactorily expressed in yes-no terms of the Boolean (two-valued) logic. A many-valued logic seems to be a well-motivated alternative in this case. This idea is not new – it has been studied already by Łukasiewicz [62] and Gödel [31]. Important basic theorems have been proved in the middle of the 20th century [16, 68]. Later on, Zadeh [118] introduced the notion of **fuzzy set** and established the foundations of **fuzzy logic** which admits the truth values to be any real numbers from the interval $[0, 1]$. This approach became wide-spread due to successful applications in automatic control [64, 114]. Fuzzy controllers as an alternative or supplement to classical controllers allowed easy solution of numerous control tasks. At present, extension of fuzzy control techniques to other areas is limited by our ability to guarantee their properties (in particular stability) in different situations. This aim requires extended theoretical background.

The study of fuzzy logics has already brought many deep theoretical results. It has been shown by Hájek in [40] that fuzzy logics offer a rich, reasonable, and justified alternative to the classical logic (as well as to some other logics studied before, e.g., the intuitionistic logic). Deep completeness theorems for various fuzzy logics have been proved there and in consequent papers [18, 25, 26, 41, 48, 49].

In parallel to this line of investigation, **MV-algebras** have been studied for years [16, 17]. They form an algebraic structure corresponding to the **Łukasiewicz logic** and this work is a permanent inspiration for fuzzy logic.

2 Types of uncertainty

Various types of uncertainty can be distinguished according to their origin. As described in [84], they can be also recognized by their relations to dependence between events.

2.1 Stochastic uncertainty

Stochastic uncertainty is typically caused by incomplete information about the initial conditions (or internal description) of the system. It assumes unlimited repeatability of measurements, each of them giving yes-no answers.

This is a standard approach which does not require a more detailed explanation. Its specifics will become clear in comparison to other alternatives.

2.2 Quantum uncertainty

Quantum uncertainty is usually caused by physical limitations of our knowledge. Repeatability of measurements is restricted because each measurement can cause an irreversible change of the state of the system. The same initial state cannot be reconstructed. Therefore it becomes impossible to observe simultaneously two events which can be observed separately. Such couples of events are called **non-compatible** or **non-commuting**.

As a consequence of quantum uncertainty, the truthfulness of some logical formulas cannot be tested, although they are composed from testable propositions. Thus, instead of a Boolean algebra, we need another, more general (non-distributive) algebraic structure to describe the event structure of such a system. The assumption that all possible measurements give yes-no answers remains valid in quantum logic.

The same situation, characterized by irreversible changes of the state during measurements, is typical also in many other fields—sociology, psychology, artificial intelligence, etc.

2.3 Fuzzy uncertainty

Fuzzy uncertainty represents the vagueness of results of an experiment. Even if the repeatability is unlimited, it may happen that the result cannot be adequately described in yes-no terms. This motivates the use of many-valued or fuzzy logic. (In this context, "classical" sets are called **crisp** or **sharp** to distinguish them from fuzzy sets.)

2.4 Distinguishing different types of uncertainty

In [84], we find that different types of uncertainty can be recognized on a simple situation: Suppose that a probability measure s attains given values on events a, b. How many real parameters (degrees of freedom) are necessary to determine s on all formulas whose atomic symbols are a, b? Different probabilities give different answers to this question. In the classical probability theory, only one degree of freedom remains (e.g., the correlation). In a quantum logic, we have (at least) one additional degree of freedom determining the probability of events which are non-zero, although their classical analogies are zero. In fuzzy logics, the (free) algebra generated by two events becomes infinite and we have infinitely many degrees of freedom. These differences may serve as additional criteria for recognition of types of uncertainty present in a given system.

2.5 Combinations of types of uncertainty

We do not consider our overview of types of uncertainty complete. Other points of view have led to different classifications. In real systems we often encounter a combination of different types of uncertainty. The usefulness of each description depends on the dominant types of uncertainty present in the system. A unified description of all the above three types of uncertainty is desirable. However, these attempts led to structures so general that it is hard to say much about their properties [71, 101]. Among these attempts, **effect algebras** (called also D-posets) seem to play an important role [23, 27].

3 Quantum logics

3.1 Quantum structures

Let us assume that we observe an atom in a box arranged as in Fig. 1. The atom can

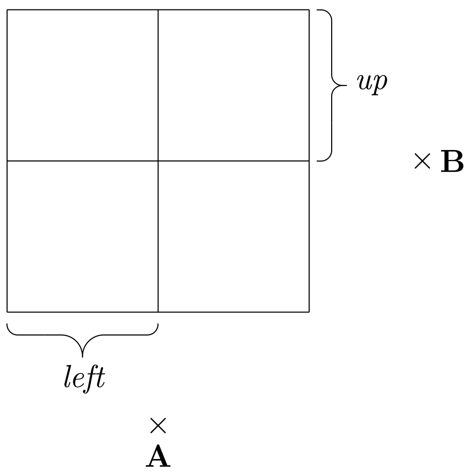


Figure 1: Experiment from Fig. 1

move between the quadrants and can emit a photon. If an observer at point **A** receives the photon, he can distinguish whether the atom was in the left or in the right half of the box. (Here we assume macroscopic dimensions of the box and ignore microscopic quantum phenomena allowing the atom to be in both halves of the box. Still we shall need a quantum model to describe the quantized radiation.) Similarly, an observer at point **B** can distinguish the upper and lower half. In the classical case, we may place two observers at points **A**, **B** and distinguish four states corresponding to the presence of the atom in particular quadrants.

In quantum systems, a simultaneous observation is impossible. Measurements are destructive (they change the state of the system irreversibly). Here a single photon can be observed only once. For the observer placed in **A**, the elementary observable events are *left*, *right*, *dark*, where *left* (resp. *right*) represents the event "the atom is observed in the left (resp. right) half", and *dark* represents the event "nothing was seen". All events observable from **A** form a Boolean algebra $A = \{\mathbf{0}, \mathbf{1}, \mathit{left}, \mathit{right}, \mathit{dark}, \mathit{left}', \mathit{right}', \mathit{dark}'\}$ (where $\mathbf{0}, \mathbf{1}$ represent the impossible and the sure event and $'$ denotes the negation). For the observer in **B**, the elementary observable events are *up*, *down*, *dark*, where *up* (resp. *down*) is the event "the atom is observed in the upper (resp. lower) half". All events observable from **B** form a Boolean algebra $B = \{\mathbf{0}, \mathbf{1}, \mathit{up}, \mathit{down}, \mathit{dark}, \mathit{up}', \mathit{down}', \mathit{dark}'\}$.

We have no tool to observe the conjunction of *left* and *up* and other events which are supposed to exist in the classical probability theory. (Here we could use the third dimension; an observer at a point outside the plane of Fig. 1 could distinguish all quadrants. Nevertheless, the example can be easily extended to a dimension which is not available.) Our system is described by two Boolean algebras, A and B. Their intersection contains their bounds $\mathbf{0}, \mathbf{1}$ and also the event *dark* and its complement (which have the same meaning for both observers – the atom did not emit any photon). All observable events are

$$L = \{\mathbf{0}, \mathbf{1}, \mathit{left}, \mathit{right}, \mathit{dark}, \mathit{up}, \mathit{down}, \mathit{left}', \mathit{right}', \mathit{dark}', \mathit{up}', \mathit{down}'\} = A \cup B.$$

They do not form a Boolean algebra but a union of Boolean algebras.

The basic structure for the description of such systems is an **orthomodular lattice** [7, 37, 39, 52]. It is a bounded lattice L (with bounds $\mathbf{0}, \mathbf{1}$ corresponding to the impossible and the sure event) with a unary operation $'\colon L \to L$ (**orthocomplementation**) such that

$$\begin{aligned} a \le b &\implies b' \le a', \\ a'' &= a, \\ a \vee a' &= \mathbf{1}, \\ a \vee b &= a \vee (a' \wedge (a \vee b)). \end{aligned}$$

(The latter equation is called the **orthomodular law**.) Every orthomodular lattice is a union of Boolean algebras [21]. Elements $a, b \in L$ are **compatible** if they are contained in a Boolean subalgebra of L. Although the lattice operations $\wedge, \vee$ are defined for any couple of elements of an orthomodular lattice, they coincide with the conjunction and the disjunction only for compatible elements. By **quantum structures** we mean not only orthomodular lattices, but also more general structures which are not lattices, orthomodular posets and orthoalgebras (see [33, 43, 97]).

Finite (and some infinite) quantum structures admit a representation by hypergraphs called **Greechie diagrams**. Vertices represent **atoms**, i.e., minimal non-zero elements. Edges represent maximal sets of mutually exclusive atoms (which correspond to maximal Boolean subalgebras). The experiment from Fig. 1 can be described by the Greechie diagram in Fig. 2. The vertex (atom) *dark* is common for both edges (Boolean subalgebras A, B).

3.2 States

A quantum state of the system can be described by a **probability measure** which is also called a **state** in this context. It is a mapping $s\colon L \to [0, 1]$ such that

$$\begin{aligned} s(\mathbf{1}) &= 1, \\ s\Big(\bigvee_{i \in \mathbb{N}} a_i\Big) &= \sum_{i \in \mathbb{N}} s(a_i) \end{aligned}$$

whenever $(a_i)_{i \in \mathbb{N}}$ is a sequence of elements which are mutually **orthogonal**, i.e., $a_i \le a_j'$ if $i \ne j$.

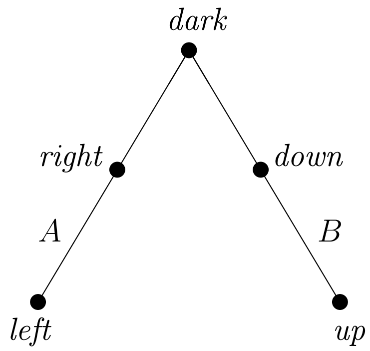


Figure 2: Greechie diagram corresponding to Fig. 1

We demonstrate it again on the experiment from Fig. 1. For an observer at **A**, the probabilities of elementary events must sum up to one,

$$s(\mathit{left}) + s(\mathit{right}) + s(\mathit{dark}) = 1.$$

Similarly, for the observer at **B**, we obtain the requirement

$$s(\mathit{up}) + s(\mathit{down}) + s(\mathit{dark}) = 1.$$

These properties can be easily seen from the Greechie diagram at Fig. 2. States on a quantum structure correspond to **states** on their Greechie diagrams, i.e., non-negative evaluations of vertices which sum up to one over each edge.

The **state space** (=the set of all states) is closed under convex combinations. Its extreme points are called **pure states**; these are states which cannot be expressed as non-trivial convex combinations of different states.

In our example, the pure states are given by the following table:

$s(\mathit{left})$	$s(\mathit{up})$	$s(\mathit{dark})$
0	0	0
0	1	0
1	0	0
1	1	0
0	0	1

All states s are uniquely determined by the values

$$s(\mathit{left}) = p,\ s(\mathit{up}) = q,\ s(\mathit{dark}) = r,$$

where $r \in [0,1]$ is arbitrary and $p, q \in [0, 1-r]$.

Notice that the orthogonality condition $a \leq b'$ in the definition of a state is strictly stronger than the usual $a \wedge b = \mathbf{0}$. For instance, $\mathit{left} \wedge \mathit{up} = \mathbf{0}$, but $\mathit{left} \not\leq \mathit{up}'$.

3.3 Different descriptions of quantum structures

Paper [85] presents a detailed introduction to quantum structures followed by new results. Its approach is rather non-standard because it takes the so-called **pasted family of Boolean algebras** as the basic structure. Each of its Boolean algebras corresponds to a maximal set of compatible elements (an edge in the Greechie diagram). Then the pasted family of Boolean algebras is associated to a unique algebraic structure, its **pasting**, equipped with a partial ordering and an orthocomplementation. We prove that the pasting results always in an **orthoalgebra**; in particular cases we obtain an **orthomodular poset** or even an **orthomodular lattice**. Conversely, each orthoalgebra can be constructed as a pasting of a pasted family of Boolean algebras. We obtain a one-to-one correspondence between orthoalgebras and pasted families of Boolean algebras (up to isomorphisms).

3.4 Representation of the state space

The above structures, including pasted families of Boolean algebras, admit to study states. We introduce a new notion of **functional isomorphism**, i.e., an affine homeomorphism between state spaces which induces a one-to-one mapping of evaluation functionals. (An **evaluation functional** φ_a induced by an element a is defined on the state space by the formula $\varphi_a(s) = s(a)$.) Further, we define **semipasted families of Boolean algebras** and states on them and we prove that every semipasted family of Boolean algebras is functionally isomorphic to some orthoalgebra, even to an orthomodular lattice. (Its pasting is a more general structure studied in [54, 100].)

A hypergraph must satisfy some conditions in order to be a Greechie diagram of a pasted family of Boolean algebras; their verification might be rather difficult.

Example 1 The hypergraph in Fig. 3a admits no states. Indeed, all its vertices can be disjointly covered by 2, resp. 3 edges, as shown in Fig. 3b, resp. c. Thus each state s has to satisfy

$$2 = \sum_{a \in V} s(a) = 3,$$

which is impossible. (The sum is taken over all vertices.)

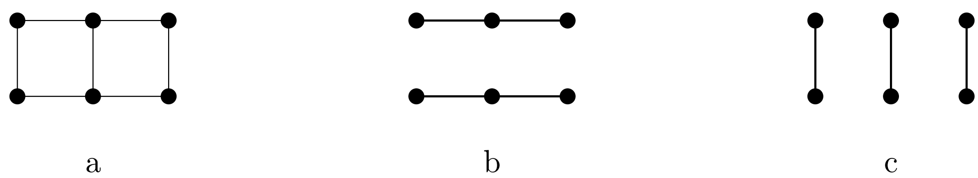
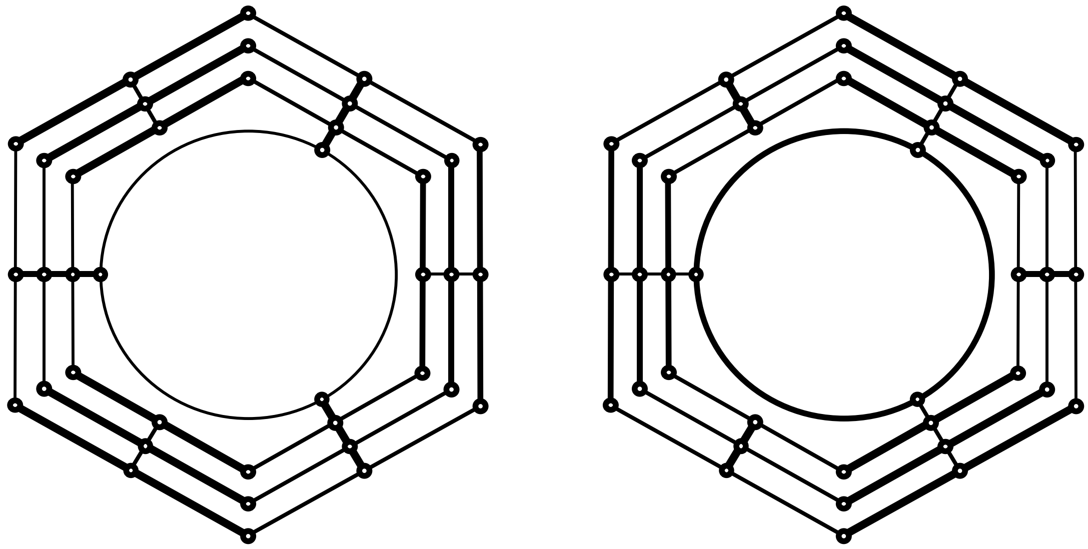


Figure 3: Hypergraph admitting no states but not corresponding to any quantum structure

However, the hypergraph in Fig. 3a does not correspond to any orthomodular lattice. For this, it is necessary to satisfy several conditions. In particular, edges with less than three vertices have to be excluded (or at least disjoint from all other edges) and also loops of order 3 and 4 are forbidden. The rectangular grid in Fig. 3 contains many loops of order 4. (See [21, 34, 76] for more details on representations of orthomodular lattices by hypergraphs.)

One of the simplest examples of an orthomodular lattice admitting no states—"the web"—is described by its Greechie diagram in Fig. 4a, resp. b. (This example is due to Rogalewicz.) All its vertices can be disjointly covered by 12, resp. 13 edges.

We have seen that it might be difficult to construct Greechie diagrams of orthomodular lattices with desired state space properties. In contrast to this, **every** hypergraph



a b

Figure 4: An OML admitting no states and many automorphisms

represents a *semipasted* family of Boolean algebras and, up to a functional isomorphism, it represents an orthomodular lattice. This gives us a very efficient tool for constructions of orthomodular lattices with given state space properties. For instance, an orthomodular lattice admitting no states can be obtained even from the simple hypergraph at Fig. 3, although it is not a Greechie diagram of any orthoalgebra. Many proofs were significantly simplified by this technique, among others the proof of Shultz's theorem which states that every compact convex set is affinely homeomorphic to the state space of some orthomodular lattice [111]. In contrast to this, the state spaces of Boolean algebras can be only simplices. The state space of our experiment from Fig. 1 is not a simplex but a pyramid (the three-dimensional convex hull of the five pure states).

3.5 The hidden variables conjecture

The quantum theory admits the existence of non-compatible events. As their conjunction cannot be tested, there is no need to assign any probability to it. Nevertheless, there still could be a classical description of a non-classical system, although it would remain unknown. This idea has led to the notion of **hidden variables** which could determine the results of quantum experiments in a classical way. Being not recognizable, they are not in direct contradiction with the limited knowledge in quantum systems. Nevertheless, the existence of hidden variables is a basic question from the philosophical and methodological point of view. Besides, it has important consequences for the choice of the mathematical model.

The idea of hidden variables was strongly defended by Einstein, Podolsky, and Rosen in [24]. (Therefore the relevant experiments are called **EPR experiments**.) This idea was rejected by Heisenberg, von Neumann, and others, but it remained a topic of discussions for several decades. The definite mathematical argument against it was the Gleason's theorem [30] which characterizes probabilities (states) on the lattice of closed subspaces of a Hilbert space. This is the principal example of a quantum structure. Linear subspaces of a Hilbert space H (in case of infinite dimension, only closed subspaces are taken) form an orthomodular lattice, $L(H)$, where

$$\begin{aligned} \mathbf{0} &= \{0\}, \\ \mathbf{1} &= H, \\ A \wedge B &= A \cap B, \\ A' &= \{x \in H \mid \forall y \in A : y \perp x\}, \\ A \vee B &= Lin(A \cup B), \end{aligned}$$

where Lin denotes the closed linear hull. Each vector $x \in H$ determines a **vector state** P_x by

$$P_x(Lin(\{y_1, \ldots, y_n\})) = \sum_{i=1}^{n} (x \cdot y_i)^2 = \sum_{i=1}^{n} \cos^2 \sphericalangle(x, y_i)$$

for any orthogonal set of vectors $y_1, \ldots, y_n \in H$. The main part of the Gleason's theorem says that the lattice of closed subspaces of a real or complex Hilbert space of a finite dimension at least 3 admits only convex combinations of vector states. As a consequence, the set of all values of a state on atoms (=one-dimensional subspaces) is convex, thus an interval. In particular, there are no two-valued states (attaining only values $0, 1$). As the two-valued states correspond to hidden variables, the Gleason's theorem gives a negative answer to the hidden variables conjecture.

When J.S. Bell heard of the Gleason's result, he was not satisfied by the complexity of its proof which, moreover, was based on another highly non-trivial mathematical theorem. He promised to find a short proof provided that the theorem holds. Bell succeeded only partially; no big simplification of the proof was found in the next years. Despite some progress in [93], the first substantial improvement—a proof of the Gleason's theorem using only elementary mathematics (but in a highly advanced way)—was published only in 1985 [19]. On the other hand, Bell succeeded to find a simple proof of the main corollary—the non-existence of hidden variables (two-valued states) [4, 5]. For this, it is sufficient to concentrate on finitely many vectors in the space. We present here an alternative proof of this fact which can be demonstrated by a few pictures.

We start with the lattice of subspaces of a three-dimensional real vector space, $\mathbb{R}^3$. Its elements are $\{0\}$, $\mathbb{R}^3$, and all one-dimensional spaces and their complements. Assume that this lattice admits a two-valued state s. Then each orthogonal basis contains exactly one vector generating a one-dimensional space on which s attains 1. For simplicity, let us associate to s a colouring of all non-zero vectors (=directions) with two colours (say, red and blue) such that each orthogonal triple contains exactly one red vector. All vectors can be represented by points on the unit sphere (because we distinguish only their directions; the norms are irrelevant). We shall prove that such a colouring does not exist.

Let us start from two red vectors, a, b, such that their angle is "large" (this choice will be discussed later), see Fig. 5a. We place a to the "north pole". All vectors orthogonal to a, b must be blue; they form two big circles on the sphere (representing two planes in the space). On these two circles, we may choose two orthogonal blue vectors. The vector orthogonal to them must be red. Thus these two vectors determine a big circle of blue vectors, see Fig. 5b. We repeat the same argument for the new big circle and one of the old ones, "the equator", see Fig. 5c. Repeating this step once more, we finally find a blue vector lying on the "northern hemisphere" at the big circle containing a, b, see Fig. 5d.

Now we distinguish two cases. In the situation in Fig. 5d, we obtained a contradiction, because one of the initial two red vectors, b, should be coloured blue. This arrangement can be achieved by an appropriate choice of orthogonal vectors if $\sphericalangle(a, b) \geq \arctan 2$.

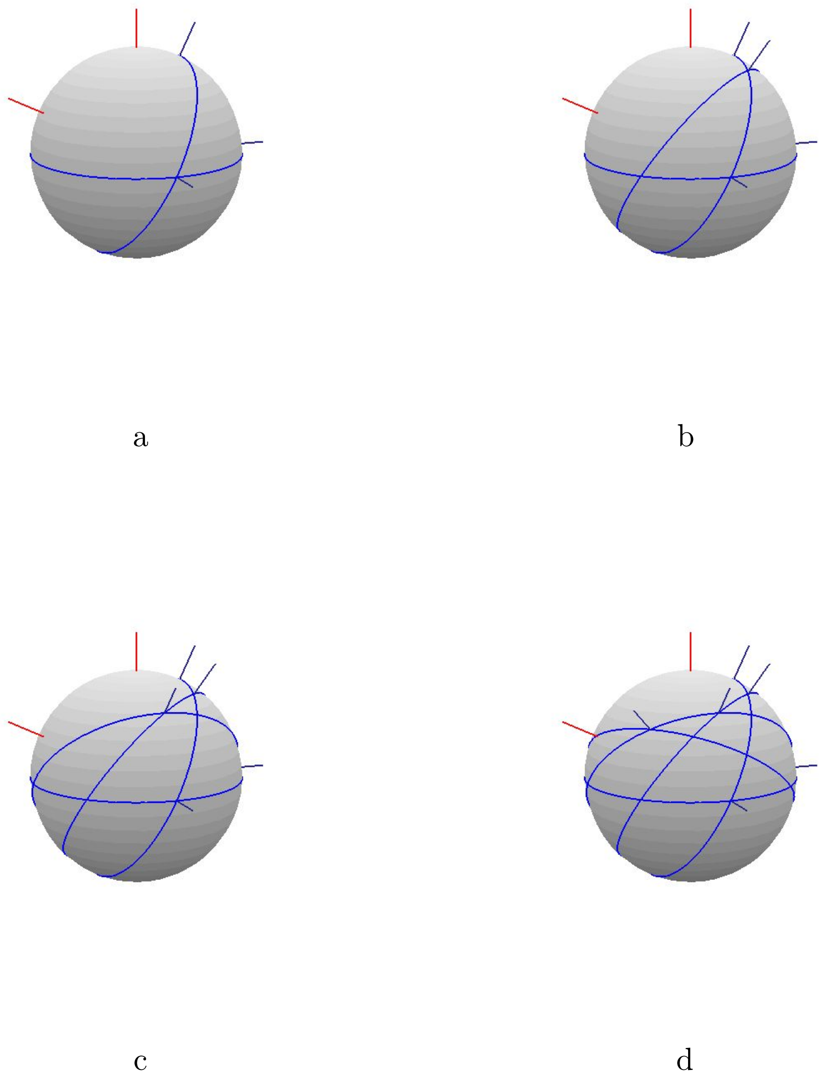


Figure 5: Vectors proving the non-existence of hidden variables

Otherwise, the blue vector is below b, but above the equator. This implies the existence of a new red vector, c, on the big circle containing a, b, such that $\sphericalangle(b, c) > \sphericalangle(a, b)$. Thus we may start the above construction with b, c instead of a, b. A routine calculation shows that after finitely many steps we obtain two red vectors with angle at least $\arctan 2$. Then the first case occurs and we again get a contradiction.

Many other arguments were brought for the non-existence of two-valued states on the lattices of subspaces of Hilbert spaces. It can be easily extended from a lower dimension to any higher finite dimension. In fact, the proof in a higher dimension can be simpler, see [69, 92, 94]. The Gleason's theorem has been generalized to infinite dimensions by Hamhalter and Pták [42], Dvurečenskij [22], and others. The prominent role of Hilbert spaces over the fields of real and complex numbers and quaternions has been explained by a famous theorem by Solèr [108].

A totally different proof of non-existence of hidden variables is due to Kochen and

Specker [59]. They have found a finite set of vectors in $\mathbb{R}^3$ which already does not admit a colouring by two colours in the above sense. The argument was improved and now the smallest known construction of this kind uses 31 vectors. Due to the originators, such results are called Kochen–Specker theorems. (Their detailed overview can be found in [69].)

A related question was posed whether there are two-valued states with values in other groups, in particular in the two-element group Z_2. We answered it in the negative in [88] for dimensions greater than 4 by a refinement of a construction due to Peres [92]. Later on, this result has been extended to dimension 4 in [44]. For dimension 3, this question is still open.

3.6 Center, automorphism group, and their independence on state space

The **center** of an orthomodular lattice is the set of all elements which are compatible to all other elements. From the physical point of view, it represents the classical part of a quantum system.

The automorphism group is another important feature of an orthomodular lattice. We have proved the independence of these attributes on the state space. To demonstrate the technique, we show it on a much simpler task of this kind.

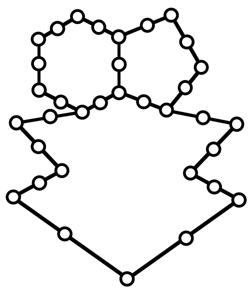


Figure 6: An orthomodular lattice admitting many states and no non-trivial automorphism

Example 2 The orthomodular lattice corresponding to the Greechie diagram in Fig. 6 admits no other automorphism than the identity because the three loops are of different orders. We connect it with the orthomodular lattice from Ex. 1 which admits no states, but many automorphisms (because of many symmetries mapping the hypergraph onto itself). We obtain the "spider in web" in Fig. 7. It is the Greechie diagram of an orthomodular lattice admitting no states because it contains the orthomodular lattice from Fig. 4 as a subalgebra. As the "body of the spider" (Fig. 6) admits no non-trivial automorphisms and it fixes the automorphisms of the whole structure, the resulting orthomodular lattice has no non-trivial automorphism. No element different from $\mathbf{0}, \mathbf{1}$ is central, so the center is trivial, too.

We obtained an example demonstrating a solution of the simplest possible case of our task.

Partial results about independence of the state space, center, and automorphism group were achieved in [9, 51, 53, 76, 75, 77, 95, 115]. These were covered by the papers [45, 78] which prove the following result:

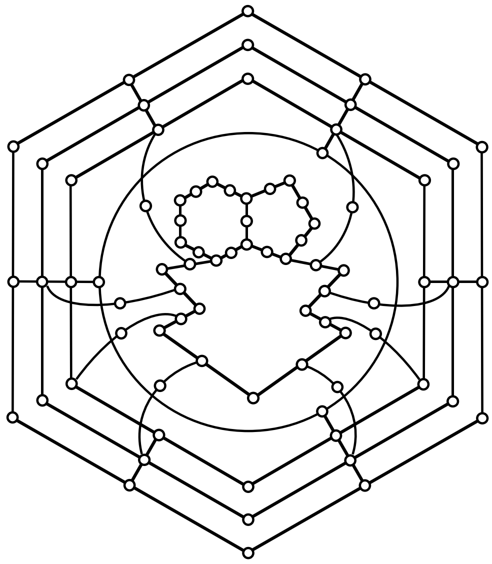


Figure 7: An orthomodular lattice admitting no states and no non-trivial automorphism

Theorem 3 *Let B be a Boolean algebra, G a group, S a compact convex space, and K an orthomodular lattice admitting at least one state. Then there is an orthomodular lattice L such that its center is isomorphic to B, its automorphism group is isomorphic to G, its state space is affinely homeomorphic to S, and K is a subalgebra of L.*

3.7 Quantum structures with rich state spaces

Many constructions of orthomodular posets were described in [76]. It extends the pasting of Boolean algebras to pastings of orthomodular posets. Among others, it introduces the so-called **substitution of an atom by an orthomodular poset**. This technique has been later used by other experts (it is called **N-R-substitution** in [61]).

The above techniques did not allow to obtain a set of states which is **rich** (also **quite full**), i.e., for each pair of incomparable events a, b there is a state s such that $s(a) = 1 > s(b)$. (For physical arguments motivating this notion, see [39].) There was a lack of combinatorial constructions resulting in structures with rich sets of states [32, 66]. A progress has been made in [67] which contributed by a construction of orthomodular lattices with rich sets of states which satisfy some linear equalities and inequalities.

This allowed to solve open problems about varieties of orthomodular lattices from [66] and [96]. In particular, it has been shown that the lattice of varieties (equational classes) of orthomodular lattices with rich state spaces has infinite width and continuum height.

3.8 Uniqueness problem for bounded observables

In the context of quantum structures, random variables have to be generalized to so-called **observables**. These are homomorphisms from the Borel σ-algebra on the real line into an orthomodular lattice. In fact, we can measure only its expectations, not particular values. A natural question formulated by Gudder [35] in 1966 is the following:

> Can two different observables on an orthomodular lattice with a rich set of states have the same expectations in all states?

This would mean that the two observables could not be distinguished by any experiment.

Without the richness assumption, a trivial positive answer is obtained from [34] or Ex. 1: There is an orthomodular lattice admitting no states. As it admits many different observables, these cannot be distinguished by their (non-existing) expectations. This example has been extended in [80] and independently in [116]: There are orthomodular lattices which do not admit non-zero measures with values in any commutative group. This result has a lot of consequences, among others it disproved the conjecture that effect algebras could be described as intervals in some commutative groups as suggested in [29]. (This effort was inspired MV-algebras – see Section 4.5, where this is possible by the use of the **Mundici functor** [17, 74].)

Despite a series of partial results, see [36, 37, 38, 98, 105, 109], the uniqueness problem for bounded observables remained open till 1995 when a positive answer has been given in [81][1].

The solution used a simplified version of a tool developed in [67] and mentioned in Section 3.7 – a construction of orthomodular lattices with rich sets of states which satisfy some equations, e.g.,

$$s(x_1) + s(x_2) + \ldots = s(y_1) + s(y_2) + \ldots$$

for some elements $x_1, y_1, x_2, y_2, \ldots$ and *all* states s.

4 Fuzzy logics

4.1 Motivation

A lot of knowledge is formulated using vague terms of a natural language. Fuzzy sets and fuzzy logic enabled to include such information in computer programs. Fuzzy control succeeded mainly because of this combination of traditional experience and new technology.

We have contributed by an analysis of the properties of Mamdani–Assilian fuzzy controllers in [73]. Looking at the rule base as a system of fuzzy relational equations (a generalization of a system of equations), we have found out that the interpretation of the rules becomes distorted by the usual inference mechanism. This becomes particularly serious when we add many rules to the base during the phase of tuning (typically if new rules are generated automatically by an algorithm). We have proposed a new inference mechanism which overcomes this difficulty and which has better mathematical properties. The enhanced controller, so-called **controller with conditionally firing rules**, has been tested on several control tasks with success. For the same rule base, it outperforms the Mamdani–Assilian controller [86].

In fact, fuzzy logic contributes to control mainly as a tool for non-linear approximation. This can be used also in other areas. We tried to combine our idea of a controller with conditionally firing rules and the Fuzzy Rule Learner (FURL) proposed by Rozich, Ioerger and Yager in [107]. The new tool was compared with other approaches in [89]. We made tests on learning of diagnostics based on medical data. Although other methods were more successful on some tasks, ours had the best performance on real data whose classification was difficult. This gives an encouraging perspective for future investigations.

In parallel, the development of statistical methods continued. They were mostly based on the classical (Kolmogorovian) probability theory. However, it is quite natural and desirable to combine these two approaches. This could allow an extension of statistical

[1]This work has been awarded by the International Quantum Structures Association in 1996

methods to events which are described by fuzzy sets and interpreted as vague statements. This seems to be against the basic paradigm of statistics, because verification/falsification requires exact criteria for the results of experiments. One may object that much experience has been collected in natural sciences, medicine, psychology, sociology, and other "soft sciences" with the use of terms which do not admit exact definitions. In particular, medical diagnosis is still often based on unsharp symptoms (elevated temperature, high blood pressure, overweight, pale skin, etc.). This motivates the effort to introduce fuzzy events to probability theory (or, vice versa, probability to fuzzy set theory).

4.2 Fuzzy logical operations

Fuzzy logic requires to extend classical logical operations from $\{0,1\}$ to the whole interval $[0,1]$ of truth values. Here we consider a **strong fuzzy negation**, i.e., a non-increasing mapping $'\colon [0,1] \to [0,1]$ such that $\alpha'' = \alpha$ for all $\alpha \in [0,1]$. Usually we restrict attention to the **standard fuzzy negation** $\alpha' = 1 - \alpha$.

The conjunction in fuzzy logic is usually interpreted by a **triangular norm** (**t-norm**), i.e., a commutative associative non-decreasing operation $\odot\colon [0,1]^2 \to [0,1]$ with neutral element 1. The basic examples are the **minimum** (the **Gödel** or **standard t-norm**)

$$\alpha \odot_{\mathbf{M}} \beta = \min(\alpha, \beta),$$

the **product t-norm**

$$\alpha \odot_{\mathbf{P}} \beta = \alpha\beta,$$

the **Łukasiewicz t-norm**

$$\alpha \odot_{\mathbf{L}} \beta = \max(\alpha + \beta - 1, 0),$$

and the Frank family of t-norms [28] defined for $\lambda \in (0,\infty) \setminus \{1\}$ by

$$\alpha \odot^{\mathbf{F}}_{\lambda} \beta = \log_{\lambda}\left(1 + \frac{(\lambda^{\alpha} - 1)(\lambda^{\beta} - 1)}{\lambda - 1}\right).$$

As limit cases, we obtain the preceding t-norms: $\odot^{\mathbf{F}}_0 = \odot_{\mathbf{M}}$, $\odot^{\mathbf{F}}_1 = \odot_{\mathbf{P}}$, $\odot^{\mathbf{F}}_{\infty} = \odot_{\mathbf{L}}$. A t-norm $\odot$ is called **strict** if it is continuous and satisfies

$$\alpha < \beta, 0 < \gamma \Longrightarrow \alpha \odot \gamma < \beta \odot \gamma.$$

The disjunction is usually taken dual to the conjunction; it is interpreted by a **triangular conorm** (**t-conorm**), i.e., a commutative associative non-decreasing operation $\oplus\colon [0,1]^2 \to [0,1]$ with neutral element 0. T-conorms $\oplus^{\mathbf{F}}_{\lambda}$ dual to the Frank t-norms $\odot^{\mathbf{F}}_{\lambda}$ (with respect to the standard negation $'$) are called **Frank t-conorms**.

Frank t-norms were found useful for introduction of a **degree of probabilistic dependence** of two events, see [84].

The interpretation of the implication is not unique in fuzzy logic. Using an analog of the classical formula

$$p \Rightarrow q = p' \vee q,$$

we obtain the so-called **S-fuzzy implication** induced by the corresponding t-norm and a fuzzy negation:

$$I_{\odot}(\alpha, \beta) = \alpha' \odot \beta.$$

More often, the **R-fuzzy implication** (**residuum**) of the t-norm $\odot$,

$$R_\odot(\alpha,\beta) = \sup\{\gamma \in [0,1] : \alpha \odot \gamma \leq \beta\},$$

is used. This choice allows to use deep results of the theory of residuated lattices and Heyting algebras. Only R-fuzzy implications give the proper evaluation of modus ponens and thus they are necessary for deduction in fuzzy logics.

All fuzzy logical operations can be extended to fuzzy subsets of some universe by the standard technique.

4.3 Tribes

An axiomatic approach imitating the classical probability theory was suggested by Höhle [47] and developed by Butnariu and Klement [13]. As an analog of a σ-algebra, they suggest a **tribe** of fuzzy sets. Let X be a non-empty set. Following [90], a **tribe** on X is a pentuplet $(\mathcal{T}, \odot, {}', 0, \leq)$, where $\mathcal{T} \subseteq [0,1]^X$, $\odot$ is a t-norm, $'$ is a fuzzy negation, 0 is the constant zero function on X, $\leq$ is the fuzzy inclusion (=the usual order of membership functions), and the following conditions are satisfied:

(T1) $0 \in \mathcal{T}$,

(T2) $f \in \mathcal{T} \Longrightarrow f' \in \mathcal{T}$,

(T3) $f, g \in \mathcal{T} \Longrightarrow f \odot g \in \mathcal{T}$,

(T4) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \Longrightarrow f \in \mathcal{T}$.

(The symbol $\nearrow$ denotes monotone increasing convergence.) The above definition is a modification of the original notion by Butnariu and Klement. They admitted only the standard negation for $'$ and, instead of (T3), (T4), they assumed

(T3+) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \Longrightarrow \bigodot_{n\in\mathbb{N}} f_n \in \mathcal{T}$.

This condition is more general, but the difference is not essential. All results by Butnariu and Klement refer to tribes which satisfy our definition, too.

When there is no risk of confusion, we speak briefly of a tribe $(\mathcal{T}, \odot)$ (as in [2]), resp. of an $\odot$**-tribe** $\mathcal{T}$ (as in [13]). Our notation follows the pattern of [117]. We also speak of an $\odot$-tribe when we need to refer to the t-norm $\odot$, but not to the tribe itself. In particular cases, when $\odot$ is the product t-norm, resp. the Łukasiewicz t-norm, we speak of a **product tribe**, resp. a **Łukasiewicz tribe**.

A **full tribe** (called a **generated tribe** in [13]) is the family of all fuzzy sets which are (as $[0,1]$-valued functions defined on the universe) measurable with respect to some σ-algebra. This σ-algebra is isomorphic to the family $C(\mathcal{T}) = \mathcal{T} \cap \{0,1\}^X$ of all crisp sets from $\mathcal{T}$.

Basic results about the structure of tribes were given by Butnariu and Klement in [12, 13]. The notion of full tribe has been generalized to a **weakly full tribe** (or **weakly generated tribe**) in [79], together with a proof that all product tribes are weakly full. This result has been generalized to all strict Frank t-norms by Mesiar [70]. Łukasiewicz tribes have a more complex structure characterized in [20, 57].

4.4 Measures on systems of fuzzy sets

We contribute to the study of σ-additive measures on σ-complete structures (provided that σ-additivity is meaningful in the context). The main difference from quantum logics is that we work with *distributive lattices* which admit a set representation. In contrast to the classical and quantum logic, we admit any truth values from $[0, 1]$.

A **probability measure** on a tribe $(\mathcal{T}, \odot, ', 0, \leq)$ is a functional $\mu\colon \mathcal{T} \to [0, 1]$ such that

(M1) $\mu(0) = 0$, $\mu(1) = 1$,

(M2) $f, g \in \mathcal{T} \Longrightarrow \mu(f \odot g) + \mu(f \oplus g) = \mu(f) + \mu(g)$, where $\oplus$ is the t-conorm dual to $\odot$,

(M3) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \Longrightarrow \mu(f_n) \to \mu(f)$,

(M4) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \searrow f \Longrightarrow \mu(f_n) \to \mu(f)$.

Conditions (M3), (M4) represent the **σ-order continuity** of μ. In preceding studies (mainly [13]), condition (M4) was not required. This resulted from an analogy with σ-algebras where (M3), (M4) are equivalent. However, without (M4) we obtain so-called **support measures** of the form

$$\mu(f) = P(\operatorname{Supp} f), \tag{1}$$

where $\operatorname{Supp} f = \{x \in X \mid f(x) > 0\}$ is the **support** of a fuzzy set f and P is a classical measure. Such measures depend only on the support and do not distinguish among positive membership degrees. This seems to be hardly motivated by applications.

Every probability measure μ w.r.t. the Łukasiewicz t-norm is a **linear integral measure**, i.e., it is of the form

$$\mu(f) = \int_X A\, dP, \tag{2}$$

where P is a classical measure on the σ-algebra $C(\mathcal{T})$. Also all σ-order continuous probability measures w.r.t. strict Frank t-norms $\odot^{\mathbf{F}}_{\lambda}$, $\lambda \in (0, \infty)$, are of this form [90]. (In [72], we have proved that without σ-order continuity every probability measure is a convex combination of a linear integral measure and a support measure.) On the other hand, these are the only strict t-norms for which formula (2) defines a probability measure on a non-Boolean tribe. Moreover, only those strict t-norms which are in some sense equivalent to Frank t-norms admit (σ-order continuous) probability measures at all. **Nearly Frank t-norms** are t-norms $\odot$ which admit an expression

$$h(\alpha \odot \beta) = h(\alpha) \odot^{\mathbf{F}}_{\lambda} h(\beta) \tag{3}$$

for some Frank t-norm $\odot^{\mathbf{F}}_{\lambda}$, $\lambda \in [0, \infty]$, and some increasing bijection $h\colon [0, 1] \to [0, 1]$ which **commutes with the standard negation**, i.e.,

$$h(\alpha') = (h(\alpha))'$$

for all $\alpha \in [0, 1]$. For a nearly Frank t-norm $\odot$ of the form (3), every probability measure is a (generally non-linear) **integral measure** of the form

$$\mu(f) = \int_X h \circ A\, dP, \tag{4}$$

where P is a classical probability measure on $C(\mathcal{T})$. (Without σ-order continuity, every probability measure is a convex combination of a support measure and an **integral measure** of the form (4), see [83].)

If $\odot$ is a strict t-norm which is not nearly Frank, then there are no (σ-order continuous) probability measures on any non-Boolean $\odot$-tribe. (Non-σ-order continuous probability measures can be only support measures in this case.)

Our results give a justification of linear integral measures of fuzzy sets as defined by (2). These have been introduced by Zadeh already in 1968 (see [119]), but without a deeper motivation. Now we see that these are (up to isomorphism) the only σ-order continuous probability measures on a tribe. However, the choice of the t-norm is not arbitrary – among strict t-norms, only nearly Frank ones are allowed. Among other t-norms, at least the Łukasiewicz t-norm admits linear integral measures.

As a side-effect, the characterizations of measures on tribes allowed to compare the semantics of fuzzy logics and extend the results of [15] in [46, 58]. In [79], we have proved that each product tribe is also an $\odot$-tribe for any measurable t-norm $\odot$. This result has been generalized to all strict Frank t-norms in [70]. In [14], we extended these results by showing that there are many other strict t-norms with this property; we have proved necessary and sufficient conditions for this to hold. On the other hand, we have found strict t-norms which do not possess this property. The principle counterexample is the **Hamacher product** [56] defined by

$$x \odot_0^{\mathbf{H}} y = \begin{cases} 0 & \text{if } x = y = 0, \\ \dfrac{xy}{x + y - xy} & \text{otherwise.} \end{cases}$$

4.5 Probability on MV-algebras

Another approach to measures of fuzzy sets is based on MV-algebras [17].

An **MV-algebra** is an algebra $A = (A, 0, \oplus, ')$, where the operation $\oplus\colon A^2 \to A$ is associative and commutative with 0 as the neutral element, the operation $'\colon A \to A$ satisfies the identities $x'' = x$ and $x \oplus 1 = 1$, where $1 = 0'$, and, in addition,

$$y \oplus (y \oplus x')' = x \oplus (x \oplus y')' \,. \tag{5}$$

Following common usage, for any elements x, y of an MV-algebra, we shall use the abbreviation $x \odot y = (x' \oplus y')'$. We shall also consider A as a distributive lattice with the ordering $x \le y \iff \exists z : x \oplus z = y$ and the corresponding lattice operations $x \vee y = x \oplus (x \oplus y')'$ and $x \wedge y = x \odot (x \odot y')'$. A **$\sigma$-complete MV-algebra** is an MV-algebra closed under countable lattice operations. Then it is also closed under $\oplus$ with countably many arguments.

As shown by Chang [16], Boolean algebras coincide with MV-algebras satisfying the equation $x \oplus x = x$. In this case the operations $\oplus, \odot$ coincide with $\vee, \wedge$, respectively.

Example 4 The real unit interval $S_\infty = [0, 1]$ equipped with the Łukasiewicz t-norm $\oplus$ and the standard fuzzy negation $'$ is an MV-algebra called the **standard MV-algebra**.

A **state** on a σ-complete MV-algebra $\mathcal{T}$ is a mapping $\mu\colon \mathcal{T} \to [0, 1]$ satisfying the following conditions:

(S1) $\mu(1) = 1$,

(S2) $f, g \in \mathcal{T},\ f \odot g = 0 \Longrightarrow \mu(f \oplus g) = \mu(f) + \mu(g)$,

(S3) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \Longrightarrow \mu(f_n) \to \mu(f)$.

The state space is a convex set; its extreme points are called **pure states**.

Due to properties of Łukasiewicz operations, the conjunction of (S1), (S2) is equivalent to the conjunction of (M1), (M2). (For strict t-norms, (S2) is much weaker than (M2).) Condition (S3) is identical to (M3) and it implies also (M4) for σ-complete MV-algebras.

Łukasiewicz tribes form a special class of MV-algebras. They are characterized in [20] as those σ-complete MV-algebras which admit a **separating set** of pure states, i.e., for each $a, b \in \mathcal{T},\ a \neq b$, there is a pure state s such that $s(a) \neq s(b)$. Although not all σ-complete MV-algebras satisfy this property, it is reasonable to require it in probability theory. (This has a Boolean analogy: Among general σ-complete Boolean algebras, only σ-algebras of subsets of a set are usually considered a good basis of a probability theory. The relation of Łukasiewicz tribes to σ-complete MV-algebras is the same as that of σ-algebras to general σ-complete Boolean algebras.)

As a consequence, the notions of state and probability measure, as introduced here, coincide for Łukasiewicz tribes. We shall see that this happens in a much more general context.

One important feature of MV-algebras (not achieved in other fuzzifications of Boolean algebras) is the existence of partitions of unity and their joint refinements. However, the coarsest joint refinement of two partitions of unity need not exist and there is no canonical formula for such refinements (see [65] for more details). Such a formula is highly desirable for probability theory. Therefore the basics of statistics (central limit theorem, etc.) were developed in [103] only for the case when $\mathcal{T}$ is an **MV-algebra with product**, i.e., a pair $(\mathcal{T}, \cdot)$, where $\mathcal{T}$ is an MV-algebra and $\cdot\colon \mathcal{T}^2 \to \mathcal{T}$ is a commutative and associative binary operation satisfying:

(P1) $1 \cdot a = a$,

(P2) $a \cdot (b \odot c') = (a \cdot b) \odot (a \cdot c)'$.

For the standard MV-algebra, the algebraic product is the only operation making it an MV-algebra with product [103]. More generally, a Łukasiewicz tribe forms an MV-algebra with product iff it is equipped with the algebraic product as $\cdot$; then we call it a **Łukasiewicz tribe with product**.

4.6 Comparison

The above two approaches coincide in the following important case:

Theorem 5 *Łukasiewicz tribes with product are exactly product tribes; states on them coincide with probability measures.*

The latter theorem holds not only for product tribes, but for all T-tribes where T is a strict Frank t-norm. What is surprising is that we obtain the same notion from two axiomatic systems where condition (S2) (formulated for the *Łukasiewicz* t-norm) is much different from condition (M2) (formulated for the *product* t-norm, resp. a strict Frank t-norm).

The overlapping of MV-algebras with product and tribes (w.r.t. strict Frank t-norms) shows that two approaches to probability on fuzzy sets converge to essentially the same notions. This opens a new field for further investigations, because some results can be directly translated from one context to the other. Among others, generalizations of the central limit theorem, laws of large numbers [103], and results about entropy can be applied to product tribes (and some other tribes), too. This applies also to recent studies of conditional probability on MV-algebras with product [50, 60] which offer a solution to an open problem from [103].

On the other hand, many results were derived for tribes, e.g. decomposition theorems [1], extensions of Lyapunov theorem [3], and applications to games with fuzzy coalitions [13]. These can be applied to MV-algebras with product (at least in the case when they are Łukasiewicz tribes).

5 Conclusions and perspectives

We contributed to basic research by characterizations of measures on various logics. Let us mention several consequences:

- Unique criteria allow to distinguish types of uncertainty present in a given system.
- A new description of state spaces on quantum structures has been developed. It simplifies many combinatorial constructions.
- We contributed to constructions of quantum structures.
- The proof of independence of several attributes of quantum structures (the state space, the center, the automorphism group) shows the necessity of additional axioms for a quantum structure to represent a reasonable physical system.
- New constructions of quantum structures with rich state spaces allowed a solution of the uniqueness problem for bounded observables formulated in 1966.
- We have proposed an enhancement of fuzzy controllers and tested it on several tasks, including medical diagnostics.
- The structure of tribes of fuzzy sets has been clarified. It allows to compare the syntax of various fuzzy logics.
- Characterization of measures on tribes allows a generalization of some classical theorems about measures on σ-algebras.
- The comparison of quantum and fuzzy logics allows better understanding of their specifics, as well as common features. It is inspiring for future research.

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WORK EXPERIENCE

Czech Technical University, Dept. of Cybernetics, Center for Machine Perception, Researcher: Responsibilities include: Research in non-standard logics and applied mathematics, lectures in Fuzzy Logic, Numerical Analysis and Computer Algebra Systems. *August, 1996, through present.*

Czech Technical University, Dept. of Mathematics, Assistant Professor: Responsibilities included: Lectures in Numerical Analysis and Computer Algebra Systems, seminars in Linear Algebra, and Mathematical Analysis, preparation and supervision of laboratories on PCs and Apple Macintosh computers, preparation of educational computer programs. *September, 1987, through July, 1996.*

Czech Technical University, Dept. of Mathematics, Research Student: Responsibilities included: Seminars in Linear Algebra, Mathematical Analysis and Theory of Probability. *September, 1983, through August, 1987.*

EDUCATION

Doctor of Science in Mathematical Logic, Academy of Sciences of the Czech Republic, February 2001.

Docent (∼ Associate Professor) in Applied Mathematics, Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University, October, 1996.

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Diploma Engineer in Technical Cybernetics (specialization Control Engineering), Department of Control Engineering, Faculty of Electrical Engineering, Czech Technical University, July, 1983.

RESEARCH

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Active participation at more than 100 international conferences.
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More than 50 papers in reviewed journals, about 100 conference papers, 3 chapters in monographs.

AWARDS

“Prize for Scientific Achievement”, awarded by the International Quantum Structures Association, 1996,
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ACTIVITIES

Member of the Editorial Board of Tatra Mountains Mathematical Publications, since 1998

International Quantum Structures Association, 1991 (member of the Council, 2001–04, member of the Nominationg Committee, 1996–2001 and since 2004)
American Mathematical Society, 1994
Association of Czech Mathematicians and Physicists, 1988
European Society for Fuzzy Logic and Technology, 1998
Czech Society for Cybernetics and Informatics, 1999 (member of the committee, since 2003)
Sisyfos (Czech Sceptics Club), 1998
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