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Czech Technical University in Praze, Faculty of Civil Engineering

Doc. RNDr. Daniela Jarušková, CSc.

Application of sequential and non-sequential change point methods to statistical process control

Sekvenční a nesekvenční odhadování změny ve stochastickém chování posloupnosti dat s aplikací v statistické kontrole jakosti

Summary

Modern engineering products, from individual components to large systems, must be designed and manufactured to be reliable in use. This has resulted in renewed emphasis on statistical techniques for designing quality into products and for identifying quality problems at various stages of production and distribution. Control charting is now used extensively in industry as a diagnostic technique for monitoring production processes to identify instability and unusual circumstances. The proper use of *statistical methods for quality control* would not be possible without good understanding of the applied procedures. The mathematical analysis of their behavior often leads to difficult mathematical problems that are solved in the scope of mathematical statistics by methods of *change point analysis*.

The change point analysis is a branch of mathematical statistics whose aim is to decide whether an observed time series is stationary or whether it changes its stochastic properties, e.g. location or spread, during observation. It has many applications and the statistical quality control belongs to the most important. Usually it is supposed that at the beginning the observed characteristics of the production process "behave correctly" according to a production design, i.e. they vary around a known target value with a known variability. We say, that the process is "in control". However, it can happen that at some unknown time point the behavior of observed variables changes due to some failures in the production process. The goal of statistical inference is to detect such a change.

Originally, the methods for statistical process control were suggested to be applied "on line", i.e., the decision, whether the process is "in control", is made after every new data is obtained. That is why the sequential methods of mathematical statistics are usually applied. As the objective is to detect a problem with the smallest possible delay, the decision rule is based on the ARL function. The ARL function expresses the mean time between a failure and its detection. However, the decision concerning the stability of a production process may be also made after the whole sequence of data is available. In this case the non – sequentional methods of mathematical statistics are to be applied. Here, the statistical inference is based on hypotheses testing. The criterion for the decision (the critical region) has to be chosen to be able to distinguish with a great reliability between the null hypothesis that claims that the process is stationary and the alternative hypothesis claiming that in an unknown time moment the process changed its stochastic behavior. Clearly, the decision rule has to be adapted to the type of change the statistician is interested in. A different rule has to be applied if one looks for a change in the mean of the process and another one if one looks for a change in the variance. The suggested rule has to take into account whether the observations are dependent or independent. The correct choice of the decision rule demands a good understanding of its properties.

The author of this lecture studied properties of the decision rules proposed for various problems under the assumption that the length of a data series is large. Especially, she studied the situations when the change occurred gradually, so that it is almost unobservable in the beginning but becomes more and more apparent later. She showed that for such a type of changes it is necessary to apply the decision rules different from the rules that are usually applied to detect a sudden change. She also studied their properties and showed that their exists a relationship between the behavior of the test statistics (corresponding to the applied critical regions) and the behavior of extremes of differentiable random processes. She applied her results to various practical problems, e.g., for detection of failures in repeated measurement of a quantity.

MSC 2000: 62P30

Souhrn

Moderní technické výrobky, ať se jedná o jednotlivé součástky či velké systémy, musí být navrženy a vyrobeny tak, aby fungovaly spolehlivě. Snaha o vysokou spolehlivost vede k opětovnému zájmu o statistické metody pro řízení jakosti a pro včasné rozpoznání problémů ve všech fázích výroby i distribuce. Kontrolní diagramy jsou stále více používány v průmyslu jako diagnostický nástroj k monitorování výrobního procesu a identifikaci jeho případné nestability způsobené poruchami. Správné použití *metod statistického řízení jakosti* není možné bez dobrého porozumění vlastností používaných postupů. Matematická analýza jejich chování často vede k řešení obtížných matematických problémů pomocí metod *analýzy bodu změny.*

Analýza bodu změny je částí matematické statistiky zabývající se problémem, zda je sledovaná časová řada stacionární nebo zda se během pozorování stochastické vlastnosti řady, jako například její poloha či rozptyl, změnily. Analýza bodu změny má mnoho aplikací, přičemž statistická kontrola jakosti patří mezi nejvýznamnější. Obvykle se předpokládá, že se na počátku měřené veličiny, které charakterizují kvalitu výrobku, chovají správně, to znamená, že kolísají kolem předem stanovené návrhové hodnoty s očekávanou variabilitou. Může se však stát, že se v nějaký neznámý časový okamžik začne chování sledovaných charakteristik měnit. To může být způsobeno poruchou ve výrobním procesu. Cílem statistické inference je odhalit takovouto změnu.

Původně byly metody statistické kontroly jakosti navrženy tak, aby mohly být používány „průběžně“, to znamená, že se rozhodnutí, zda je proces ještě „pod kontrolou“, provádí znovu a znovu s každým novým měřením. Z toho důvodu se zde obvykle používají sekvenční metody matematické statistiky, jejichž cílem je odhalit problém s co možná nejmenším zpožděním. Proto je rozhodovací pravidlo založeno na ARL funkci. ARL funkce vyjadřuje průměrnou dobu mezi okamžikem, kdy porucha nastala, a časem, kdy byla detekována. V jiných případech se ale rozhodnutí o stabilitě výrobního procesu provádí až poté, co byla získána všechna data. Zde se obvykle používají nesekvenční metody matematické statistiky a rozhodnutí se zakládá na výsledku testování hypotéz. Rozhodovací kritérium (obvykle ve formě kritického oboru) musí být vybráno tak, aby rozlišilo co možná nejspolehlivěji, zda platí nulová hypotéza, která tvrdí, že je sledovaný proces stacionární, nebo zda platí alternativa, která tvrdí, že se proces v nějakém neznámém časovém okamžiku změnil. Navíc je zřejmé, že rozhodovací pravidlo musí být navrženo tak, aby odhalilo právě ten typ změny, na který zaměřujeme svoji pozornost. V případě, že se zajímáme o změnu ve střední hodnotě, musíme použít jiné pravidlo, než v případě, když nás zajímá změna v rozptylu. Při tvorbě rozhodovacího pravidlo musíme brát v úvahu, zda jsou pozorované veličiny závislé či nikoliv. Správný výběr pravidla vyžaduje, abychom dobře rozuměli jeho vlastnostem.

Autorka této přednášky studovala vlastnosti rozhodovacích pravidel navržených pro nejrůznější typy problémů za předpokladu, že získaná posloupnost dat je velmi dlouhá. Speciálně se věnovala situaci, kdy ke změně dochází postupně. K postupné změně dochází tehdy, jestliže na začátku je změna tak nepatrná, že se téměř nedá rozpoznat, ale časem se stává stále více a více výraznější. Ukázala, že pro odhalení postupných změn je třeba použít pravidla odlišná od pravidel pro detekci náhlé změny. Studovala rovněž jejich vlastnosti a ukázala, že existuje vztah mezi chováním testové statistiky, na jejímž základě je vytvořen kritický obor, a chováním maxima diferencovatelného náhodného procesu. Získané výsledky použila pro různé praktické problémy, mimo jiné například pro detekci chyb v opakovaném měření určité veličiny.

Klíčová slova: statistická kontrola jakosti, detekce bodu změny, změna polohy, změna rozptýlenosti, sekvenční přístup, kontrolní diagramy, Shewhartův diagram, CUSUM diagram, EWMA diagram, ARL funkce, nesekvenční přístup, testování hypotéz, testová statistika maximálního typu, asymptotické rozdělení, náhodné procesy.

Keywords: statistical quality control, change point detection, change in location, change in variance, sequential approach, control charts, Shewhart chart, CUSUM chart, EWMA chart, ARL function, non-sequential approach, hypotheses testing, test statistic of maximum type, asymptotic distribution, random processes.

České vysoké učení technické v Praze

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Autor:	Doc. RNDr. Daniela Jarušková, CSc.
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1. Introduction

Improvement of the products quality should be the main goal of modern industry. A very important role in the process leading to the better quality is played by measurement. It is necessary to have some data measuring the quality of an output to decide whether and how the quality might be improved. In almost all the situations where the data exhibit random fluctuation statistical methods are to be applied. The aim of the statistical process control procedures is to separate variation that we ordinarily expect of a process, from that which may appear due to special or assignable causes.

Statistical process control and allied techniques of sampling inspection and quality control are exactly 80 years old. In May 1924, Walter A. Shewhart of Bell Telephone Laboratories developed the first sketch of a modern control chart. The statistical process control was used extensively in World War II, but lost its importance as industries converted to peacetime production. It was in Japan where after World War II the philosophy that good quality leads to greater productivity was broadly accepted. The great emphasis on quality was one of the principal causes for the after - war Japanese industrial miracle. Since then interest in the statistical process control rose and fell down again. Nevertheless, there are many examples of how important and successful the statistical process control may be. Courses of statistical methods for the statistical process control became part of curriculum at many universities and polytechnics.

A useful class of methods that may be applied to the statistical process control are *change - point detection methods*. The change - point detection methods are statistical methods that enable to detect changes in stochastic model of studied data. It is usually supposed that at successive time points $t_1 < t_2 < \dots$ a quantity (or several quantities) is measured so that a process is observed. In the statistical process control the measured quantities are chosen to characterize quality of a product. The aim of the change - point methods is to decide whether the observed process is stable or whether at some unknown time point it changes its properties.

Application of change - point detection methods to the stochastic process control is usually easy. The idea of many suggested procedures is clear and can be understood by any undergraduate student. On the other hand when studying of stochastic properties that characterize the procedures one comes to non-trivial mathematical problems. Many of these problems were solved successfully in the past, others have been solved rather recently and many are still open.

2. Sequential and non-sequential methods

We start with a simple situation. A product is manufacturing continually and a certain characteristic of its quality is measured successively at time points $0 < t_1 < t_2 < \dots$ This characteristic may be hardness of a plastic, tensile strength, water content in a solution, weight of a powder packed in a capsule etc. In this way we are obtaining a sequence of observations $X_1, X_2, \dots$ It is supposed that at the beginning the variables "behave correctly" according to a production design, i.e. they vary around a known value, which is called the "target value". Very often it is supposed that the variance is known from the past experience with the production process. We say that at the beginning the process is *"in control"*. Now, it can happen that at some unknown moment ν the process changes, e.g., its mean abruptly or gradually changes its value, and the process becomes *"out of control"*. The moment ν is called *change - point*.

The aim of statistical inference is to decide whether a change in the behavior of the process occurred. The decision can be made sequentially, i.e. after every new data is obtained, with the goal to discover any possible change as soon as possible and take appropriate actions. The other approach is to wait until a whole sequence of n observations $X_1, \dots, X_n$ is available and try to decide with the greatest possible reliability, in the other words to apply rules that keep the probability of erroneous decision small. Methods suitable for the first type of decision are called *sequential* and procedures based on them are called *on-line*, while methods suitable for the second type of decision are called *non-sequential* and respective procedures are called *off-line*.

3. Criterion for on-line statistical process control

The efficient decision rules for on-line stochastic process control detect a change with a small delay on one side and on the other side the intervals between false alarms are long. This idea is explained in mathematical language in the following lines.

For simplicity, we assume that random variables $X_1, X_2, \dots$ have absolutely continuous distribution functions. Moreover, we suppose that *before the change point* the variables are distributed according to the density function f_0 while *after the change point* according to the another density function, say f_1. The density f_0 is supposed to be known. As to the density f_1, we discuss situations when it is both known and unknown. Throughout our applications and examples we often assume, for greater transparency of the text, that both the f_0 and f_1 are normal.

Let us denote by P_0 the distribution under which $X_1, X_2, \dots$ are independent identically distributed (iid) random variables with the density function f_0 and by $\{P_\nu,\ \nu = 1, 2, \dots\}$ the distribution under which $X_1, \dots, X_{\nu-1}$ are iid with the density function f_0 and $X_\nu, X_{\nu+1}, \dots$ are iid with the density function f_1. By $\mathsf{E}_\nu,\ \nu = 0, 1, 2, \dots$ we denote the corresponding expectations. Our main aim is to find a stopping time τ such that if change occurs at time ν then the delay for its detection $(\tau - \nu)^+$ would be small. A reasonable measure of "quickness of detection" of change occurring at time ν is the smallest number C_ν such that for all realizations $x_1, \dots, x_{\nu-1}$ of $X_1, \dots, X_{\nu-1}$ and $\tau \geq \nu$

$$\mathsf{E}_\nu\Big(\tau - \nu + 1 \mid X_1 = x_1, \dots, X_{\nu-1} = x_{\nu-1}\Big) \leq C_\nu$$

holds. As a kind of the *worst case criterion*, let us define $\overline{\mathsf{E}}\,\tau = \sup_{\nu \geq 1} C_\nu$. The decision to have small $\overline{\mathsf{E}}\,\tau$ must be, of course, balanced against the need to have a controlled frequency of *false reactions*. In other words, when there is no change then τ would be large, hopefully infinite. It was shown, however, that in order to have $\overline{\mathsf{E}}\tau$ finite it is necessary that τ has a finite expectation even under P_0. An appropriate type of restrictions on false reactions is therefore

$$\mathsf{E}_0\tau \geq B,$$

where the constant B is to be prescribed.

4. Basic on-line procedures

In this work we deal with three methods based on different stopping times (stopping rules):
1) *Shewhart algorithm (Shewhart chart, Shewhart procedure)*

$$\tau = \inf \left\{ n \,\middle|\, \log \frac{f_1(X_n)}{f_0(X_n)} \geq h_1 \right\}.$$

2) *CUSUM algorithm (CUSUM procedure)*

$$\tau = \inf \left\{ n \,\middle|\, \widetilde{S}_n - \min_{0 \leq j \leq n} \widetilde{S}_j \geq h_2 \right\},$$

where

$$\widetilde{S}_n = \sum_{i=1}^{n} \log \frac{f_1(X_i)}{f_0(X_i)}, \qquad \widetilde{S}_0 = 0.$$

3) *Exponentially weighted moving average algorithm*

$$\tau = \inf \left\{ n \,\middle|\, \overline{X}_{EWMA}(n) \geq h_3 \right\},$$

where

$$\overline{X}_{EWMA}(n) = (1-\lambda)\overline{X}_{EWMA}(n-1) + \lambda X_n, \qquad 0 < \lambda \leq 1.$$

4.1. Shewhart procedure

The Shewhart procedure is the oldest and the best known procedure of all sequential quality control algorithms. As noticed above, the stopping time has the form

$$\tau = \inf \left\{ n \,\middle|\, \log \frac{f_1(X_n)}{f_0(X_n)} \geq h \right\}.$$

To decide how to find the critical limit h it is necessary to evaluate $\mathsf{E}_0\tau$ and $\overline{\mathsf{E}}\,\tau$. As the stopping time τ for the Shewhart procedure is independent of the past of the process, the distribution of the delay does not depend upon the place where the change occurs so that $\overline{\mathsf{E}}\,\tau = \mathsf{E}_1\tau$. Provided P represents an arbitrary probability under which $X_1, X_2, \ldots$ are iid, the distribution of τ is geometric, i. e. for $j = 1, 2, \ldots$

$$P(\tau = j) = \left(1-p\right)^{j-1} p, \quad p = P\left(\log \frac{f_1(X_1)}{f_0(X_1)} \geq h \right), \tag{1}$$

so that $\mathsf{E}\tau = p^{-1}$. Especially, (1) holds for P_0 and P_1.

4.1.1. Normal distribution – one-sided Shewhart chart for shift in mean

In this section we will study independent random variables $X_1, X_2, \ldots$ distributed according to a normal distribution with a known variance σ^2. (Further $\Phi(\cdot)$ denotes the distribution function and $\phi(\cdot)$ denotes the density of the standard normal distribution.) Let us denote by P_μ the distribution under which they are iid according to $N(\mu, \sigma^2)$. We suppose that the process is *in control* if the mean $\mu = 0$ and is *out of control* if the mean $\mu = \mu^* > 0$. The stopping time $\tau = \tau_+$, which depends on the parameter h (respectively b), has the form

$$\tau_+ = \inf \left\{ n \,\middle|\, \frac{1}{\sigma^2} \left(\mu^* X_n - \frac{\mu^{*2}}{2} \right) \geq h \right\} = \inf \left\{ n \,\middle|\, X_n \geq \frac{h\sigma^2}{\mu^*} + \frac{\mu^*}{2} \right\} =$$
$$= \inf \left\{ n \,\middle|\, X_n \geq b \right\}.$$

To find the optimal value of b we will study the relationship between b and $\mathsf{E}_0\tau_+$, $\mathsf{E}_{\mu^*}\tau_+$ respectively. Under P_0, the stopping time τ_+ has the geometric distribution

$$P_0(\tau_+ = j) = (1-p_0)^{j-1}p_0, \qquad j = 1, 2, \dots,$$

where

$$p_0 = P_0\big(X_n \geq b\big) = 1 - \Phi\big(b/\sigma\big),$$

so that

$$\mathsf{E}_0\tau_+ = \frac{1}{p_0} = \frac{1}{1-\Phi\big(b/\sigma\big)}. \tag{2}$$

Under P_{μ^*}, the stopping time τ_+ has again the geometric distribution and

$$\mathsf{E}_{\mu^*}\tau_+ = \frac{1}{p_{\mu^*}} = \frac{1}{P_{\mu^*}\big(X_n \geq b\big)} = \frac{1}{1-\Phi\left(\frac{b-\mu^*}{\sigma}\right)}.$$

The usual rule for finding the value of b is to pick a desired value of $\mathsf{E}_0\tau_+$ and to solve (2) (w.r.t. b). Let us notice that in this case the choice of b, which gives τ_+ unambiguously, is independent upon the value μ^*. Therefore, we can use the same τ_+ for any $\mu > 0$. It is evident that the mean delay interval between the change and its detection is:

$$\mathsf{E}_\mu\tau_+ = \frac{1}{1-\Phi\left(\frac{b-\mu}{\sigma}\right)}.$$

In the statistical process control a crucial notion is the *average run length (ARL) function* which (in the sequential analysis) plays similar role as the power function does in the hypotheses testing. In our particular case it is the function (in μ) given by

$$ARL_+(\mu) = \mathsf{E}_\mu\tau_+$$

with one parameter b.

In Table 1 the values of the ARL function of the one-sided Shewhart procedure for detecting a positive shift in the mean of the normal distribution, $b = 3\,\sigma$, are given.

	μ					
	$-\sigma$	$-\sigma/2$	0	$\sigma/2$	σ	$3\sigma/2$
b=3σ	31 574	4299	740.8	161.04	43.96	14.97
	2σ	$5\sigma/2$	3σ	$7\sigma/2$	4σ	$9/2\sigma$
b=3σ	6.30	3.24	2.00	1.45	1.19	1.07

Table 1. Selected values of ARL function of one-sided Shewhart procedure for detecting a positive shift in mean of normal distribution, $b = 3\,\sigma$.

Analogously, we can consider the situation that the process is *in control* if the mean $\mu = 0$ and is *out of control* if the mean $\mu = \mu^* < 0$. The stopping time $\tau = \tau_-$, which depends on the parameter h, resp. b, has the form

$$\tau_- = \inf\left\{n \,\middle|\, \frac{1}{\sigma^2}\left(\mu^* X_n - \frac{\mu^{*2}}{2}\right) \geq h\right\} = \inf\left\{n \,\middle|\, X_n \leq -b\right\}.$$

The corresponding $ARL_-(\mu)$ function can be found similarly as $ARL_+(\mu)$.

4.1.2. Normal distribution – two-sided Shewhart chart for shift in mean

Let us suppose that the observations $X_1, X_2, \ldots$ are independent normally distributed with a known variance σ^2. Let the process be *in control* if the mean $\mu = 0$ and be *out of control* if the mean $\mu \neq 0$. Let us define the stopping time of a two-sided symmetric Shewhart chart $\tau = \min\{\tau_+, \tau_-\}$, where

$$\tau_+ = \inf\Big\{n \mid X_n \geq b\Big\} \quad \text{and} \quad \tau_- = \inf\Big\{n \mid X_n \leq -b\Big\}.$$

The mean interval between false reactions is

$$\mathsf{E}_0\tau = \frac{1}{p_0} = \frac{1}{P_0\big(|X_n| \geq b\big)} = \frac{1}{2\big(1 - \Phi(b/\sigma)\big)}. \tag{3}$$

Similarly as in the one-sided case, the usual rule for the choice of b is to pick a desired value of $\mathsf{E}_0\tau$ and to solve the equation (3). The stopping time τ is again determined only by the value of b so that we can use the same stopping algorithm for all $\mu \neq 0$.

The average run length function is of the form

$$ARL(\mu) = \mathsf{E}_\mu\tau = \frac{1}{1 - \Big(\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{-b-\mu}{\sigma}\right)\Big)}.$$

Suppose that independent normally distributed random variables with known variance σ^2 are observed. The target value is zero. We would like to detect an arbitrary shift in the mean. We wish that the mean interval between false reactions $\mathsf{E}_0\tau = 370$. The problem is to find the appropriate constant b. According to (3), the constant b must satisfy

$$2\big(1 - \Phi(b/\sigma)\big) = \frac{1}{370} \quad \equiv \quad \frac{b}{\sigma} = \Phi^{-1}\left(1 - \frac{1}{740}\right) \quad \equiv \quad b = \sigma\Phi^{-1}\big(1 - 0.00135\big),$$

so that $b \approx 3\,\sigma$, i.e. we stop the procedure if $|X_n| \geq 3\,\sigma$. The mean delay in detecting, for example the shift $\mu = \sigma$, equals

$$\mathsf{E}_\mu\tau \approx 44.$$

In Table 2 the values of the ARL function of the two-sided Shewhart procedure for detecting an arbitrary shift in the mean of normal distribution, $b = 3\,\sigma$, are given. The $ARL(\mu)$ function is plotted in Figure 1.

	μ				
	0	$\pm 0.5\,\sigma$	$\pm 1\,\sigma$	$\pm 1.5\,\sigma$	$\pm 2\,\sigma$
b=3σ	370	155.22	43.89	14.97	6.30
	$\pm 2.5\,\sigma$	$\pm 3\,\sigma$	$\pm 3.5\,\sigma$	$\pm 4\,\sigma$	
b=3σ	3.24	2.00	1.45	1.19	

Table 2. Selected values of ARL function of two-sided Shewhart procedure for detecting arbitrary shift in mean of normal distribution, $b = 3\,\sigma$.

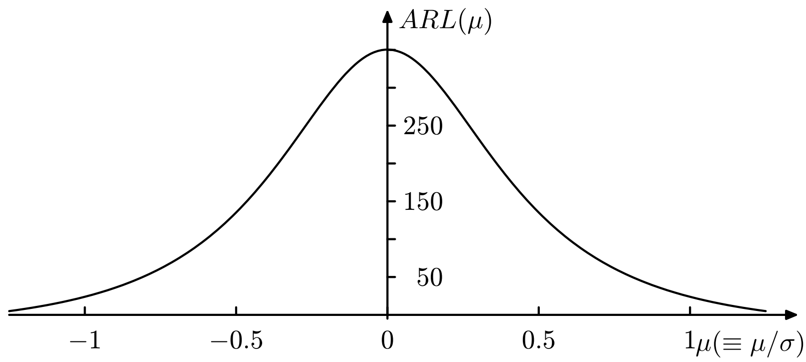

Figure 1. $ARL(\mu)$ function of two-sided Shewhart procedure for detecting arbitrary shift in mean of normal distribution, $b = 3\,\sigma$.

4.1.3. Shewhart chart for control of average level

As we saw, the important characteristic for determining *control limits* is the ARL function. However, many authors follow a long tradition and recommend, as a "rule of thumb", to take *action control limits* at the distance $3\,\sigma$ from the target value. Moreover, the usually applied Shewhart chart contains in addition to the action limits also so - called *warning limits* at the distance $2\,\sigma$ from the target value. Wetherill & Brown (1991) recommend to take an action if either one point is outside the action limits or two successive points are outside the same warning limits.

In applications of the Shewhhart procedure for the shift in the mean, instead of working with the single variables $X_1, X_2, \ldots$, one usually deals with the averages of a small number (say $m = 3 \approx 5$) of consecutive variables, i.e. with

$$\overline{X}_1 = \frac{1}{m}\sum_{i=1}^{m} X_i,\ \overline{X}_2 = \frac{1}{m}\sum_{i=m+1}^{2m} X_i, \ldots$$

The averaging reduces the variance, because if the variance of X_i's is σ^2 then the variance of $\overline{X}_i$'s is $\sigma_m^2 = \sigma^2/m$.

4.2. CUSUM procedure

CUSUM procedure is closely related to the following problem of the hypotheses testing. Suppose that $X_1, \ldots, X_n$ (n fixed for a moment) have been observed. We consider the problem of testing the null hypothesis that $X_1, \ldots, X_n$ are distributed according to the same known density function f_0 against the alternative that, at unknown time ν, $\nu \leq n$, the change occurred so that $X_1, \ldots, X_{\nu-1}$ are iid with the density function f_0 while $X_\nu, \ldots, X_n$ are iid with another *known* density function f_1. If we knew the time ν we could use the log-likelihood ratio

$$\widetilde{S}_n - \widetilde{S}_{\nu-1} = \sum_{i=\nu}^{n} \log \frac{f_1(X_i)}{f_0(X_i)}$$

as the test statistic and reject the null hypothesis if $\widetilde{S}_n - \widetilde{S}_{\nu-1} \geq K$, K being an appropriate constant. If the time of change is unknown, we reject the null hypothesis if

$$\widetilde{g}_n = \max_{0\leq j\leq n}\left(\widetilde{S}_n - \widetilde{S}_j\right) = \widetilde{S}_n - \min_{0\leq j\leq n} \widetilde{S}_j \geq \widetilde{h}.$$

Notice that $\widetilde{g}_n$, $n \geq 1$, can be defined also by the following recursion

$$\widetilde{g}_n = \max\left(\widetilde{g}_{n-1} + \log\frac{f_1(X_n)}{f_0(X_n)},\ 0\right), \qquad \widetilde{g}_0 = 0.$$

Let us turn back to the situation described in Section 3., i.e., that we have an infinite sequence of observations $X_1, X_2, \dots$ available. An intuitively appealing stopping rule is

$$\tau = \inf\left\{n \mid \widetilde{g}_n \geq \widetilde{h}\right\}.$$

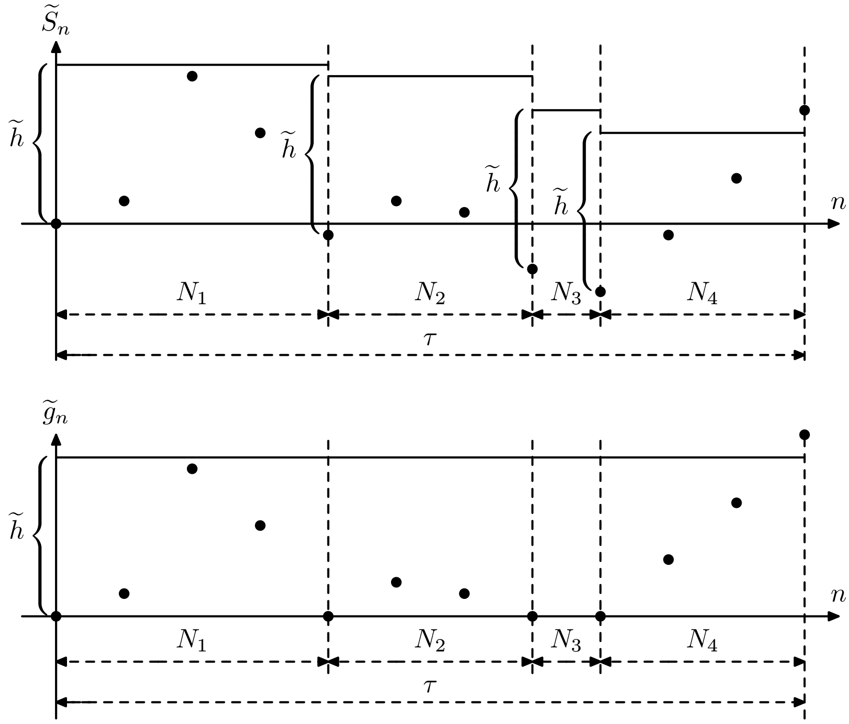

Figure 2. Example of behavior of sequences $\left\{\widetilde{S}_n, \widetilde{g}_n\right\}$.

Let us notice that $\widetilde{g}_n$ measures the current height of the random walk $\widetilde{S}_j,\ j = 0, 1, 2, \dots$ above its minimal value. Whenever the random walk establishes a new minimum, i.e. $\widetilde{S}_n = \min_{0 \leq k \leq n} \widetilde{S}_k$, the process forgets its past and starts again in the sense that for all $j \geq 0$

$$\widetilde{S}_{n+j} - \min_{0 \leq k \leq n+j} \widetilde{S}_k = \widetilde{S}_{n+j} - \widetilde{S}_n - \min_{0 \leq k \leq j}\left(\widetilde{S}_{n+k} - \widetilde{S}_n\right).$$

The sequence $\widetilde{g}_n$ is in fact the random walk with one reflecting boundary.

We will study the properties of $\left\{\widetilde{g}_n\right\}$ to evaluate $\mathsf{E}_0\tau$ and $\overline{\mathsf{E}}\tau$, 3.. At first we show that $\mathsf{E}_1\tau = \overline{\mathsf{E}}\tau$. The inequality $\mathsf{E}_1\tau \leq \overline{\mathsf{E}}\tau$ is obvious. The inequality $\mathsf{E}_1\tau \geq \overline{\mathsf{E}}\tau$ results from the following argument. The observations $X_1 = x_1, \dots, X_n = x_n$ determine $\widetilde{g}_n = t \geq 0$ (depending on $X_1, \dots, X_n$). Since $X_{n+1}, \dots$ are independent upon the previous X_i's, the sequence $\widetilde{g}_{n+1}, \widetilde{g}_{n+2}, \dots$ behaves just as $\widetilde{g}_1, \widetilde{g}_2, \dots$ would have once started with $\widetilde{g}_0 = t \geq 0$. Since this last assertion would make any succeeding $\widetilde{g}$'s smaller, it would not increase the time required to reach $\widetilde{h}$.

Hence we have to find how $\mathsf{E}_0\tau$ and $\mathsf{E}_1\tau$ depend on $\widetilde{h}$. $\mathsf{E}_0\tau$ is the expectation of τ if all $X_1, X_2, \dots$ are distributed according to the density f_0 and denotes the average time interval between the false reactions if the process is *in control*. Similarly, $\mathsf{E}_1\tau$ is the expectation of τ if all $X_1, X_2, \dots$ are distributed according to the density f_1 and denotes the average delay between the change and its detection if the process is *out of control*.

To evaluate $\mathsf{E}_0\tau$ and $\mathsf{E}_1\tau$ we can use the results from the sequential analysis as the *CUSUM* procedure is in fact a sequence of Wald's sequential tests with the (log) limits $\{0, \widetilde{h}\}$. If the sum of the log-likelihood ratios is greater than or equal to $\widetilde{h}$ then the procedure definitively ends. If it is smaller than or equal to zero then a (new) sequential test begins using only the new observations. This idea can be expressed exactly as follows. First let us recall that $\tau = \inf\left\{n \,|\, \widetilde{g}_n \geq \widetilde{h}\right\}$ and introduce

$$N = N_1 = \inf\left\{n \geq 1 \,\middle|\, \widetilde{S}_n \notin \left(0, \widetilde{h}\right)\right\}.$$

If $\widetilde{S}_{N_1} \geq \widetilde{h}$ then $\tau = N_1$. Otherwise

$$\widetilde{S}_{N_1} = \min_{0 \leq k \leq N_1} \widetilde{S}_k$$

and we put

$$N_2 = \inf\left\{n \geq 1 \,\middle|\, \widetilde{S}_{N_1+n} - \widetilde{S}_{N_1} \notin \left(0, \widetilde{h}\right)\right\}.$$

If $\widetilde{S}_{N_1+N_2} - \widetilde{S}_{N_1} \geq \widetilde{h}$ then $\tau = N_1 + N_2$. Otherwise $\widetilde{S}_{N_1+N_2} \leq \widetilde{S}_{N_1}$ and

$$\widetilde{S}_{N_1+N_2} = \min_{0 \leq k \leq N_1+N_2} \widetilde{S}_k.$$

In general, let

$$N_k = \inf\left\{n \geq 1 \,\middle|\, \widetilde{S}_{N_1+\cdots+N_{k-1}+n} - \widetilde{S}_{N_1+\cdots+N_{k-1}} \notin \left(0, \widetilde{h}\right)\right\}.$$

It is easy to see that

$$\tau = N_1 + \cdots + N_M$$

where

$$M = \inf\left\{k \,\middle|\, \widetilde{S}_{N_1+\cdots+N_k} - \widetilde{S}_{N_1+\cdots+N_{k-1}} \geq \widetilde{h}\right\}.$$

For illustration see Figure 2.

Wald's identity yields

$$\mathsf{E}_i\tau = \mathsf{E}_i N \cdot \mathsf{E}_i M, \qquad i = 0, 1.$$

The random variable M is under P_0, as well as under P_1, geometrically distributed with

$$\mathsf{E}_i M = \frac{1}{P_i\left(\widetilde{S}_N \geq \widetilde{h}\right)}, \qquad i = 0, 1,$$

hence

$$\mathsf{E}_i\tau = \frac{\mathsf{E}_i N}{P_i\left(\widetilde{S}_N \geq \widetilde{h}\right)}, \qquad i = 0, 1.$$

To evaluate how $\mathsf{E}_0\tau$ depends on $\widetilde{h}$, we must find $\mathsf{E}_0 N$ and $P_0\left(\widetilde{S}_N \geq \widetilde{h}\right)$. Analogously, to evaluate how $\mathsf{E}_1\tau$ depends on $\widetilde{h}$ we must find $\mathsf{E}_1 N$ and $P_1(\widetilde{S}_N \geq \widetilde{h})$.

The method for finding the relationship between $\widetilde{h}$ and $\mathsf{E}_0\tau$, resp. $\mathsf{E}_1\tau$, is based on the Fredholm integral equations. Let us denote for $i = 1, 2$,

$$\mathsf{P}_i(z) = P_i\left\{\widetilde{S}_N \leq 0 \,\middle|\, \widetilde{S}_0 = z\right\},$$
$$\mathsf{N}_i(z) = \mathsf{E}_i\left\{N \,\middle|\, \widetilde{S}_0 = z\right\}.$$

Then

$$\mathsf{E}_i\tau = \frac{\mathsf{N}_i(0)}{1-\mathsf{P}_i(0)}, \qquad i=1,2,$$

and the functions $\mathsf{P}_i(z)$ and $\mathsf{N}_i(z)$, $i=0,1$, satisfy the equations

$$\mathsf{P}_i(z) = \int_{-\infty}^{0} g_i(y-z)\,dy + \int_0^{\widetilde{h}} \mathsf{P}_i(y)\,g_i(y-z)\,dy,$$

$$\mathsf{N}_i(z) = 1 + \int_0^{\widetilde{h}} \mathsf{N}_i(y)\,g_i(y-z)\,dy,$$

where $g_i(y)$ is the density of $Y = \log\big(f_1(X_1)/f_0(X_1)\big)$ under the probability $P_i,\ i=1,2.$

4.3. Normal distribution – one-sided CUSUM procedure for shift in mean

Here we apply the above considerations to the case of independent normally distributed random variables $X_1, X_2, \ldots$ with known variance σ^2. Let us denote again by P_μ the distribution under which they are iid according to $N(\mu,\sigma^2)$. The process is *in-control* if the mean $\mu=0$ and is *out-of-control* if the mean $\mu=\mu^*>0$. Applying the notation and the results covering the *CUSUM* technique (summarized in the previous lines) in the way that the density of $N(0,\sigma^2)$ stands for f_0 and the density of $N(\mu,\sigma^2)$ for f_1. Then the log-likelihood ratio $\widetilde{S}_n$ is of the form

$$\widetilde{S}_n = \sum_{i=1}^{n}\left(\mu^* X_i - \frac{\mu^{*2}}{2}\right)\frac{1}{\sigma^2} = \sum_{i=1}^{n}\left(X_i - \frac{\mu^*}{2}\right)\frac{\mu^*}{\sigma^2}.$$

Let us denote for $\mu^*>0$

$$k = \frac{\mu^*}{2}\ \ldots \text{ the reference value,} \quad h = \widetilde{h}\,\frac{\sigma^2}{\mu^*}\ \ldots \text{ the threshold value,}$$

and

$$S_n(k) = \sum_{i=1}^{n}\big(X_i - k\big), \qquad S_n = S_n(0) = \sum_{i=1}^{n} X_i.$$

The *CUSUM* statistics $\{\widetilde{g}_n\}$ can be expressed in the form

$$\widetilde{g}_n = \max_{0\le j\le n}\left(\widetilde{S}_n - \widetilde{S}_j\right) = \max_{0\le j\le n}\frac{\mu^*}{\sigma^2}\sum_{i=j+1}^{n}\left(X_i - \frac{\mu^*}{2}\right) = \frac{\mu^*}{\sigma^2}\max_{0\le j\le n}\Big(S_n(k) - S_j(k)\Big)$$

and the stopping time is

$$\tau_+ = \inf\left\{n \;\middle|\; \max_{0\le j\le n}\frac{\mu^*}{\sigma^2}\sum_{i=j+1}^{n}\left(X_i - \frac{\mu^*}{2}\right) \ge \widetilde{h}\right\} = \inf\left\{n \;\middle|\; \max_{0\le j\le n}\sum_{i=j+1}^{n}\big(X_i - k\big) \ge h\right\}. \tag{4}$$

There exists another commonly used parameterization of the problem which is more geometrically oriented and can be introduced through the new constants θ and d defined by

$$\theta = \operatorname{arctg}\, k, \quad d = \frac{h}{k}\cdot$$

Using this parameterization we can rewrite the stopping time as

$$\tau_+ = \inf\left\{ n \,\middle|\, \max_{0\le j\le n}\Big(S_n - S_j - (n-j)\,\mathrm{tg}\;\theta\Big) \ge d\,\mathrm{tg}\,\theta \right\}\cdot$$

The validity of this inequality can be easily derived using the lower arm of so called V-mask, see Figure 3.

Let us summarize that the stopping time τ_+ characterizing the *CUSUM* procedure is determined by one of the equivalent couples of parameters, i. e. (k, h) or (d, θ).

Applying the general *CUSUM* scheme explained at the beginning of this chapter we are able to evaluate $\mathsf{E}_0\tau_+$ and $\mathsf{E}_{\mu^*}\tau_+$ for fixed μ^*, where

$$k = \frac{\mu^*}{2} > 0 \qquad \text{and} \qquad h = \widetilde{h}\,\frac{\sigma^2}{\mu^*}\cdot$$

However, we might be interested in the calculation of the average run length function $ARL_+(\mu) = \mathsf{E}_\mu\tau_+$ for arbitrary fixed parameters $\big(k, h\big)$. We shall touch this problem in the following lines.

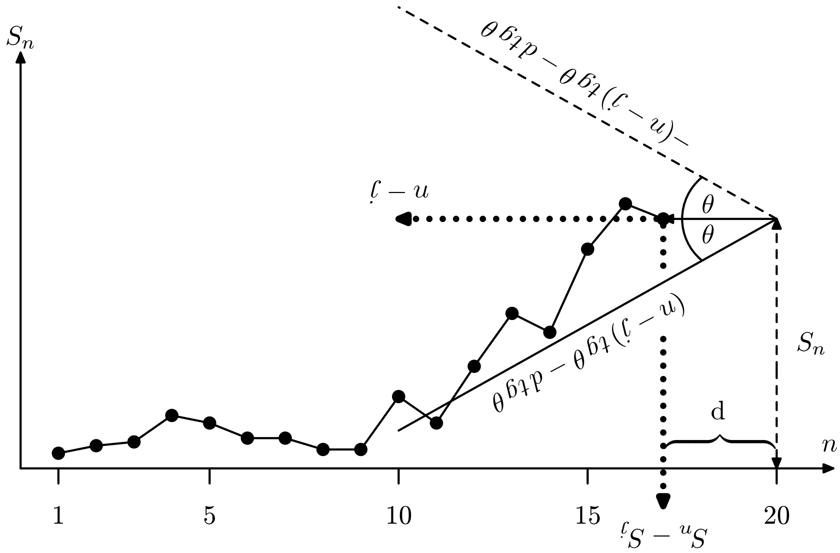

Figure 3. V-mask.

It is clear that the method of integral equations can be used also to calculate the expectation of τ_+ under an arbitrary distribution P_μ. Without loss of generality we can suppose that $\sigma^2 = 1$. The average run length function

$$\mathrm{ARL}_+(\mu) = \mathsf{E}_\mu\tau_+ = \frac{\mathsf{N}_\mu(0)}{1 - \mathsf{P}_\mu(0)},$$

where $\mathsf{P}_\mu(z)$ and $\mathsf{N}_\mu(z)$ satisfy the integral equations

$$\mathsf{P}_\mu(z) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y - z - (\mu - k))^2}{2}\right\} dy + \\ + \int_0^h \mathsf{P}_\mu(y)\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y - z - (\mu - k))^2}{2}\right\} dy,$$

and

$$\mathsf{N}_\mu(z) = 1 + \int_0^h \mathsf{N}_\mu(y) \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\big(y - z - (\mu - k)\big)^2}{2} \right\} dy.$$

The usual recommendations for the choice of k and h of $CUSUM$ procedure are as follows:

- Choose k such that $\mu = 2k$ is the value of the process where it becomes important to detect shift of given magnitude;
- Pick a desired mean time between false reactions ($MTBFR$) and choose, for already fixed k, such h that $ARL_+(0) = \mathsf{E}_0\tau_+ = MTBFR$.

Suppose that independent normally distributed random variables with the variance σ^2 are observed. The target value is zero. We would like to detect a positive shift in the mean provided the mean interval between false reactions $\mathsf{E}_0\tau_+ = 740$. Values of the ARL function for selected values of the parameters k, h and $\sigma = 1$ are summarized in the following table.

		μ					
h	k	-1	-0.5	0	0.5	1	1.5
8.006	0.25	10^9	838807	740	28.77	11.34	7.02
4.766	0.50	10^7	71032	740	35.19	9.87	5.43
2.487	1.00	276883	12757	740	67.85	13.34	5.36
1.538	1.50	59677	6237	740	108.50	22.48	7.31
1.006	2.00	25366	4162	740	148.49	35.83	11.21
h	k	2	2.5	3	3.5	4	4.5
8.006	0.25	5.08	3.98	3.27	2.77	2.41	2.13
4.766	0.50	3.73	2.84	2.29	1.92	1.65	1.45
2.487	1.00	3.15	2.21	1.70	1.38	1.16	1.00
1.538	1.50	3.54	2.21	1.58	1.23	1.00	0.85
1.006	2.00	4.72	2.57	1.68	1.23	0.96	0.79

Table 3. Values of ARL function of one-sided $CUSUM$ procedure for detecting a positive shift in mean of normal distribution with $\sigma^2 = 1$ for selected values of parameters k, h.

Let us suppose that it is for us important to detect the shift of the size $\mu = \sigma$. In that case we stop the process as soon as

$$\max_{0 \le j \le n} \sum_{i=j+1}^{n} \left(\frac{X_i}{\sigma} - 0.5 \right) > 4.766.$$

Using this stopping rule, the mean delay in detecting $\mu = \sigma$ is approximately 10 observations.

If in reality the shift would be of size $2\,\sigma$, while our procedure is based on assumptions $\mathsf{E}_0\,\tau_+ = 740$, and a desire is to detect (preferably) the shift of the size $\mu = \sigma$, we get immediately (see Table 3) that the mean delay to detect such a shift is 3.73 observations (in mean).

On the other hand, provided $\mathsf{E}_0\tau_+ = 740$ and we want to detect the shift $\mu = 2\,\sigma$ preferably, we should use $(h,k) = (1, 2.487)$. The mean delay to detect $\mu = 2\sigma$ would be then 3.15 observations.

For the negative shift, the stopping time has the form $\big(k, h > 0\big)$

$$\tau_- = \inf\left\{ n \,\middle|\, \min_{0\le j\le n} \sum_{i=j+1}^{n} \big(X_i + k\big) \le -h \right\},$$

respectively for the parameters θ and d

$$\tau_- = \inf\left\{ n \,\middle|\, \min_{0\le j\le n} \Big(S_n - S_j + (n-j)\operatorname{tg}\,\theta\Big) \le -d\operatorname{tg}\theta \right\}. \tag{5}$$

The validity of this inequality can be discovered using the upper arm of the V-mask. The average run length function for negative alternative with the stopping time τ_- can be found analogously as for τ_+.

4.4. Normal distribution – two-sided CUSUM procedure for arbitrary shift in mean

A two-sided symmetric *CUSUM* procedure for detecting an arbitrary shift in mean from the target value $\mu = 0$ with the parameters k and h is given by the stopping time

$$\tau = \min\,\big(\tau_+, \tau_-\big),$$

where τ_+ and τ_- were defined by (4), resp. by (5).
For discovering the validity of these equations we can use symmetrical V-mask with $\theta = \operatorname{arctg}\,k$ and $d = h/k$.

It was shown by Kemp (1961) that

$$\frac{1}{\mathsf{E}_\mu\tau} = \frac{1}{\mathsf{E}_\mu\tau_+} + \frac{1}{\mathsf{E}_\mu\tau_-},$$

so that for $ARL(\mu) = \mathsf{E}_\mu\tau,\ ARL_+(\mu) = \mathsf{E}_\mu\tau_+,\ ARL_-(\mu) = \mathsf{E}_\mu\tau_-$ and

$$ARL(\mu) = \frac{ARL_+(\mu)ARL_-(\mu)}{ARL_+(\mu) + ARL_-(\mu)}$$

holds.

The usual recommendations for the choice of k and h are as follows:

1. Choose k such that $\mu = |2k|$ is the magnitude of the shift which we wish to detect.

2. Pick a desired mean time between false reactions ($MTBFR$) and choose, for already fixed k, such h that $ARL(0) = \mathsf{E}_0\tau = MTBFR$.

Suppose that independent normally distributed random variables with variance $\sigma^2 = 1$ are observed and the target value is zero. We would like to detect an arbitrary shift in the mean provided the mean interval between false reactions $\mathsf{E}_0\tau = 370$. Values of the ARL function for selected values of the parameters k, h are summarized in the following table.

		μ				
h	k	0	± 0.5	± 1	± 1.5	± 2
8.006	0.25	370	28.77	11.34	7.02	5.08
4.766	0.50	370	35.17	9.87	5.43	3.73
2.487	1.00	370	67.49	13.34	5.36	3.15
1.546	1.50	370	106.64	22.48	7.31	3.54
1.006	2.00	370	143.38	35.78	11.21	4.72
h	k	0	± 2.5	± 3	± 3.5	± 4
8.006	0.25	370	3.98	3.27	2.77	2.41
4.766	0.50	370	2.84	2.29	1.92	1.65
2.487	1.00	370	2.21	1.70	1.38	1.16
1.546	1.50	370	2.21	1.58	1.23	1.00
1.006	2.00	370	2.57	1.68	1.23	0.96

Table 4. Values of the ARL function of two-sided $CUSUM$ procedure for detecting an arbitrary shift in mean of normal distribution with $\sigma^2 = 1$ for selected values of parameters k, h.

If the variance σ^2 is known but different from 1 we can proceed analogously as in the previous example.

5. Exponentially weighted moving average

The *Exponentially Weighted Moving Average* algorithm ($EWMA$) is usually applied if we wish to detect the change in the mean of the variables $\{X_i\}$ with common density f. Supposing the target value is zero and the variance σ^2 is known, the exponentially weighted moving average is defined by the recursion

$$\overline{X}_{EWMA}(n) = (1-\lambda)\overline{X}_{EWMA}(n-1) + \lambda X_n, \qquad \lambda \in (0,1],\ n = 1, 2, \ldots$$

For discovering a positive shift one sided algorithm raises alarm if

$$\overline{X}_{EWMA}(n) \geq h,$$

and the two-sided algorithm raises alarm if

$$\left|\, \overline{X}_{EWMA}(n) \,\right| \geq h.$$

The average run length function ARL has two parameters h and λ. The second constant λ is a smoothing constant. If $\lambda = 1$ then we get the Shewhart chart. On the other hand, if λ is small the $EWMA$ procedure resembles $CUSUM$.

Crowder (1987a) used Fredholm integral equation whose derivation is based on the same idea as the one for integral equation for $CUSUM$ procedure. If $L(u)$ is ARL of two sided procedure, given that the weighted moving average starts in u $\left(\overline{X}_{EWMA}(0) = u\right)$, then

$$
\begin{aligned}
L(u) &= 1.P\Big(\big|\,(1-\lambda)u + \lambda X_1\,\big| > h\Big) + \\
&+ \int_{\{|(1-\lambda)u+\lambda y|\le h\}} \Big(1 + L((1-\lambda)u + \lambda y)\Big)\, f(y)\, dy = \\
&= 1 + \frac{1}{\lambda}\int_{-h}^{h} L(y)\, f\left(\frac{y-(1-\lambda)u}{\lambda}\right)\, dy.
\end{aligned}
$$

If $L(u)$ is ARL of the one-sided procedure for the detection of a positive shift, given that the weighted moving average starts in u, then $L(u)$ satisfies

$$
L(u) = 1 + \frac{1}{\lambda}\int_{-\infty}^{h} L(y)\, f\left(\frac{y-(1-\lambda)u}{\lambda}\right)\, dy.
$$

Usually the control limits for the exponentially weighted moving average are specified to be $h = L\sigma_{EWMA}$, where

$$
\sigma^2_{EWMA} = \left(\frac{\lambda}{2-\lambda}\,\sigma^2\right).
$$

Let us suppose that the observations are normally distributed with the unit variance and let the target value be zero. We would like to detect a positive as well as negative shift in the mean and we wish that the mean interval between false reactions $\mathsf{E}_0\tau = ARL(0) = 370$. Several values of ARL function are given in the following table.

	μ				
	0	± 0.5	± 1	± 1.5	± 2
$L = 2.50,\quad \lambda = 0.50$	370	26.6	10.8	6.8	5
$L = 2.75,\quad \lambda = 0.12$	370	29.6	9.6	5.6	4
	± 2.5	± 3	± 3.5	± 4	
$L = 2.50,\quad \lambda = 0.50$	4.0	3.4	2.9	2.6	
$L = 2.75,\quad \lambda = 0.12$	3.2	2.6	2.3	2.0	

Table 5. Values of the ARL function of two-sided $EWMA$ procedure for detection of arbitrary shift in the mean of normal distribution with $\sigma^2 = 1$ for selected values of L, λ, μ (reprinted from Crowder (1987a)).

6. Remarks on application of change-point detection methods to on-line statistical process control

In the preceding sections we explained how to apply sequential change - point detection methods to several techniques of on-line process control. We have shown that if we desire to control performance of these techniques their parameters have to be set according to some criterion. In practice the most often applied criterion is based on the ARL function that expresses the mean time until the process is stopped. Some statisticians object that in some cases other characteristics of run length (e.g. quantiles) might be of interest.

The control techniques we described above were illustrated for detection of a change in the mean of the process. The engineers know that the change of mean is very often accompanied by a change of variance. In practice the Shewhart chart for mean is always used together with the Shewhart chart for range and standard deviation. Similarly, CUSUM chart to detect a change in variance always accompanies CUSUM chart for a change in mean.

In simple textbooks it is usually supposed that the observed variables $X_1, X_2, \ldots$ are distributed according to a normal distribution. The described methods can be adapted to the case where the observed variables are distributed according to a distribution from so - called Koopman - Darmois family, see Antoch and al. (2001).

It is known that correlation between successive measurements affects markedly the performance of the Shewhart charts but the effect is even larger for CUSUM and EVMA charts. The adaptation of these techniques to dependent data was suggested, e.g., by Nikiforov (1983).

The described statistical process control methods deal with one variable at a time, whereas in fact most practical situation are multivariate. A lot of work has been done for on-line detection of a change in parameters of linear or general linear regression. However, the interaction among variables that affect quality of a product may be very complex. A general approach that would cover all the situations will be probably never developed. It seems more likely that every special problem will require a special treatment.

7. Example

The data $\{X_1, \ldots, X_{250}\}$ presented by Figure 4 were simulated in the following way:

$$\begin{aligned} X_1, \ldots, X_{90} &\sim N(0,1) \\ X_{91}, \ldots, X_{250} &\sim N(0.5,1) \end{aligned} \tag{6}$$

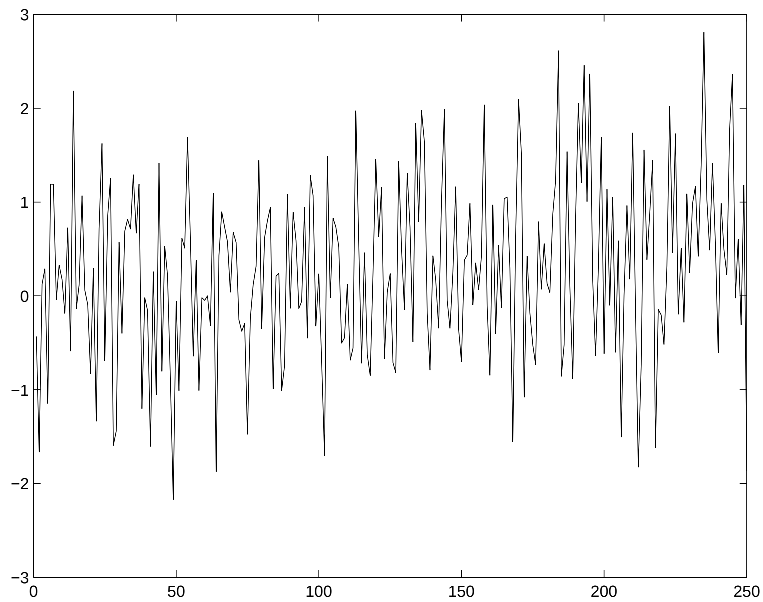

Figure 4. Simulated data according to (6)

Clearly, the shift of the size 0.5 is very small and it can be hardly seen by a naked eye. Therefore, detection of such a small shift is rather difficult.

First, we apply the Shewhart control chart for average level where averages of 5 successive data were considered. The control lines are ± 1.341 and the warning lines ± 0.894. The change was detected after we observed $39 \times 5 = 195$ observations, see Figure 5.

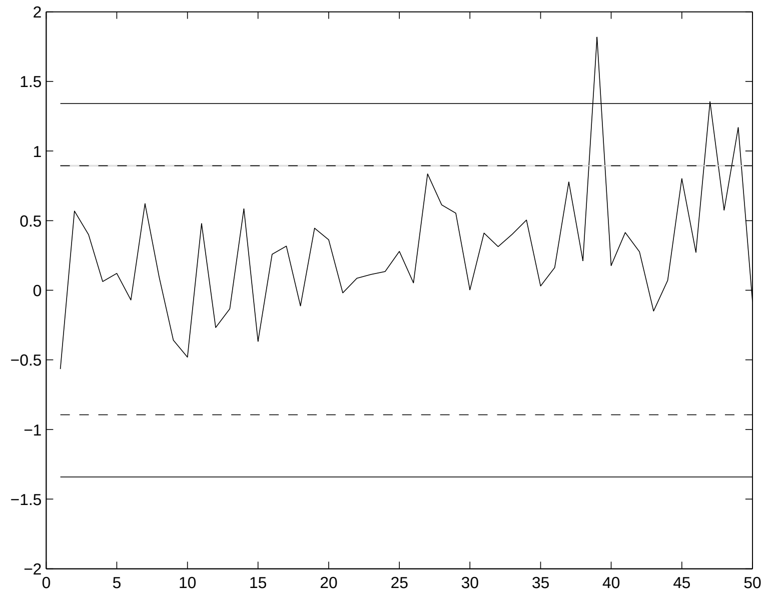

Figure 5. Shewhart chart with warning (dashed lines) and control limits (solid lines) for averages from 5 observations.

The CUSUM chart with $k = 0.25$ and $h = 8$ detected the change after $n = 137$ observations were obtained. Figure 6 shows that the CUSUM chart has not yet detected a change after 130^{th} observation but it has detected a change after 137^{th} observation. However, we have to keep in mind that we have applied here the CUSUM chart optimal for a shift 0.5. Moreover, CUSUM charts are known to perform well for small size changes. Figure 7 shows how the CUSUM chart looks like for the whole batch of $n = 250$ data.

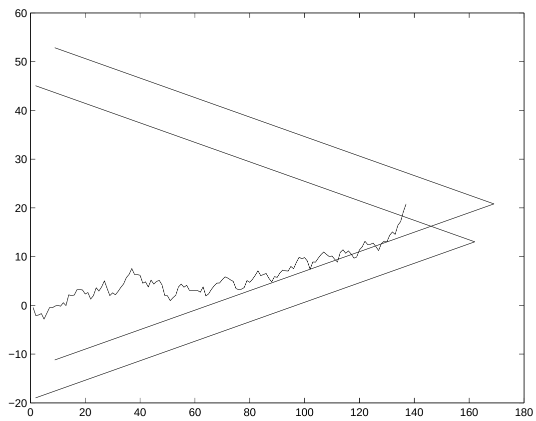

Figure 6. Application of CUSUM chart after 130^{th} and 137^{th} observation.

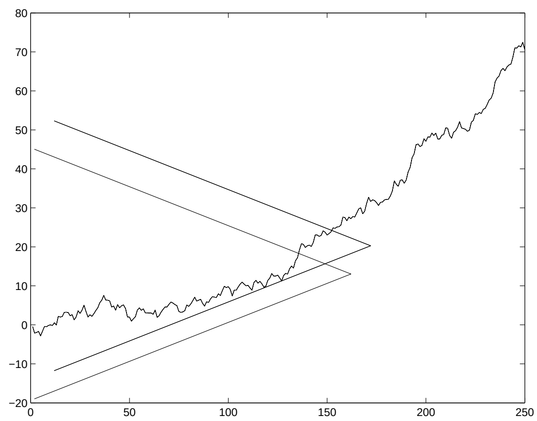

Figure 7. CUSUM chart for all 250 observations.

Figures 8 and 9 shows an application of EWMA charts with $\lambda = 0.5$ and $L = 2.50$. Since $\sigma^2 = 1$ then $h = 1.44$. It is interesting to notice that the change was again detected after the 137^{th} observation.

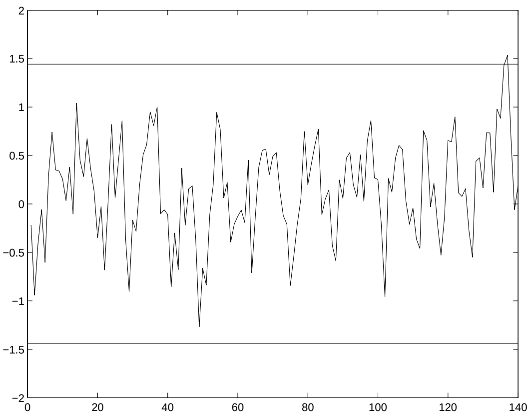

Figure 8. Application of EWMA chart for the first 140 observations with $h = 1.44$

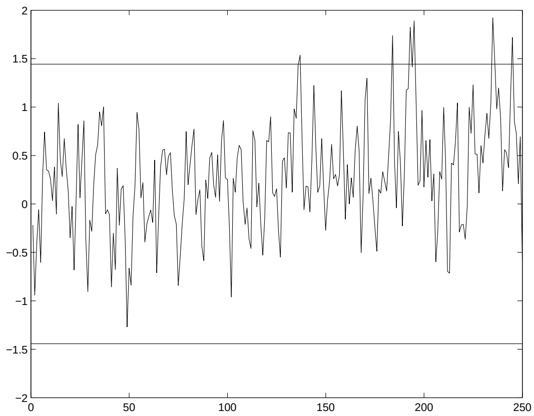

Figure 9. EWMA chart for all 250 observations with $h = 1.44$.

8. Application of non-sequential change - point methods to off-line statistical process control

In the off-line statistical process control the decision, whether the observed series $X_1, X_2, \ldots, X_n$ is stationary or whether a change of a specific kind occurred, is not taken sequentially with each new observation but only when all the n observations are available. This decision is usually based on hypotheses testing. *The null hypothesis claims that the process is stationary while the alternative hypothesis claims that the process is non-stationary and the stationarity was violated in a specific way.*

We start again with the simplest case of detecting a change in the mean of observed variables. We suppose that at the beginning the process in in control, i.e., it varies around a known (target) value μ_0 with a known variance σ_0^2. As the μ_0 and σ_0^2 are known, the observed variables may be standardized to have a zero mean and unit variance so that without any loss of generality we can set $\mu_0 = 0$ and $\sigma_0^2 = 1$. The objective of statistical inference is to decide whether at some unknown time moment the process changed its behavior and started to vary around a different value a. For simplicity we suppose that the variance remained the same. The null hypothesis H and the alternative A may be set as follows:

$$
\begin{aligned}
&H : X_i = e_i, && i = 1, \ldots, n, \\
&A : \exists\, k \in \{0, \ldots, n-1\} \quad \text{such that} && \\
&\qquad X_i = e_i, && i = 1, \ldots, k, \\
&\qquad X_i = a + e_i, && i = k+1, \ldots, n,
\end{aligned}
\tag{7}
$$

where $a \neq 0$ and $\{e_i\}$ are independent identically distributed (iid) random variables (errors).

9. Decision rules based on hypotheses testing

In the classical theory of hypotheses testing it is supposed that the sample space, i.e. the space of all possible results $X_1 = x_1, \ldots, X_n = x_n$, is divided into two regions S_0 and S_1. The set S_0 is called the *region of acceptance* and the set S_1 the *region of rejection* or *critical region.* The critical region may often be expressed in the form $\{\boldsymbol{x}; T(\boldsymbol{x}) > C\}$, where $T(\boldsymbol{x})$ is a function of the observed values $\boldsymbol{x} = (x_1, \ldots, x_n)$ and C is an appropriate constant. The variable $T(X_1, \ldots, X_n)$ is called a *test statistic* and the constant C a *critical value.* If the above inequality is fulfilled then the null hypothesis is rejected, otherwise it is not rejected. If the number of observations n is given then the decision rule consists in the choice of the test statistic $T(\boldsymbol{x})$ and the critical value C. It is desired that the constant C is chosen in such a way that under the null hypothesis the inequality $T(\boldsymbol{x}) > C$ is fulfilled with a small given probability, which is called the *significance level.* If the probability that $T(\boldsymbol{x}) > C$ is only bounded by the significance level then the critical value C is called *conservative.*

There exist several procedures for deriving test statistics for tests with large *power*, i.e., when the null hypothesis is rejected with a large probability, if the alternative is true. One of the most popular is *the likelihood ratio method.*

We illustrate the likelihood ratio method on a simple case when the change point is known and equal to k, the shift size a is arbitrary unknown constant and the variables $\{X_i\}$ are normally distributed. For testing the null hypothesis H against the alternative A:

$$\begin{aligned} H:\ & X_i = e_i, && i = 1, \ldots, n, \\ A:\ & X_i = e_i, && i = 1, \ldots, k, \\ & X_i = a + e_i, && i = k+1, \ldots, n, \end{aligned} \tag{8}$$

the log - likelihood ratio Λ_k has the form:

$$\begin{aligned} \Lambda_k &= \sup_a \log \frac{\prod_{i=1}^n f_A(X_i)}{\prod_{i=1}^n f_H(X_i)} = \sup_a \log \frac{\prod_{i=1}^k \phi(X_i) \prod_{i=k+1}^n \phi(X_i - a)}{\prod_{i=1}^n \phi(X_i)} \\ &= \sup_a \left\{ -\frac{1}{2} \sum_{i=k+1}^n (X_i - a)^2 + \frac{1}{2} \sum_{i=k+1}^n X_i^2 \right\} = \frac{1}{2(n-k)} \left(\sum_{i=k+1}^n X_i \right)^2 . \end{aligned}$$

The null hypothesis H is rejected when $\Lambda_k > C_\alpha$, which can be equivalently expressed as

$$\left| \frac{1}{\sqrt{n-k}} \sum_{i=k+1}^n X_i \right| > \sqrt{2C_\alpha},$$

where C_α is a constant chosen so as to correspond to the fixed significance level α. In other words, the log-likelihood ratio is a function of the average of the second part of the series of observations $\{X_i\}$. As the normality is assumed the critical value $\sqrt{2C_\alpha} = u_{1-\alpha/2}$, where $u_{1-\alpha/2}$ is the $100(1-\alpha/2)\%$ quantile of $N(0,1)$.

9.1. Detecting a change in mean

When the change point is unknown (so that both a and k are unknown), we have to take the supremum of the log-likelihood ratio with respect to both of them, i.e.,

$$\max_{0 \le k \le n-1} \sup_a \log \frac{\prod_{i=1}^k \phi(X_i) \prod_{i=k+1}^n \phi(X_i - a)}{\prod_{i=1}^n \phi(X_i)} = \max_{0 \le k \le n-1} \frac{1}{2(n-k)} \left(\sum_{i=k+1}^n X_i \right)^2 .$$

The test statistic usually applied for the case with the unknown change point is of the form

$$\max_{0\le k\le n-1}\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right|\right\}. \tag{9}$$

The statistic (9) belongs to the so - called *maximum - type test statistics.* The null hypothesis H is rejected if for a suitably chosen constant $C_{1\alpha}$

$$\max_{0\le k\le n-1}\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right|\right\} > C_{1\alpha}.$$

The constabt $C_{1\alpha}$ is chosen so as to correspond to the fixed significance level α. This rule is reasonable because it rejects H if for at least one k, $0\le k\le n-1$,

$$\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right| > C_{1\alpha}.$$

Similarly, it can be shown that for the one-sided alternative with $a>0$, the null hypothesis is rejected if

$$\max_{0\le k\le n-1}\left\{\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right\} > C_{2\alpha},$$

where $C_{2\alpha}$ is again an appropriately chosen constant.

For decision about rejection of the null hypothesis H we need again to know critical values of the suggested test statistics. Clearly, it holds

$$P\left(\max_{0\le k\le n-1}\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right|\right\} > x\right) \ge P\left(\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right| > x\right).$$

Therefore, the $100\,\alpha$%-quantile of the distribution of statistic (9) is larger than $u_{1-\alpha/2}$. Analyzing the same data set, we reject the null hypothesis in the case when the change point is known much more often than in the case of an unknown change point.

To find the exact distribution of (9) means to find the distribution of the maximum of absolute values of standardized normal variables that are (unfortunately) correlated. The correlation coefficients are

$$corr\left(\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i, \frac{1}{\sqrt{n-l}}\sum_{i=l+1}^{n}X_i\right) = \sqrt{\frac{n-l}{n-k}}, \quad k\le l.$$

Theoretically, it should not be a problem to find the distribution of (9). However, in practice the distribution is so complex, that its quantiles (desired critical values) may be computed only for small values of n, see Hawkins (1977).

Sometimes, the approximate critical values may be quite satisfactory for practical use. To find approximate critical values we can use a very simple idea by applying the Bonferroni inequality as follows:

$$P\left(\max_{0\le k\le n-1}\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right|\right\} > C\right) = P\left(\bigcup_{k=0}^{n-1}\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right| > C\right\}\right)$$
$$\le \sum_{k=0}^{n-1}P\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n}X_i\right| > C\right\} = n\,P\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_i\right| > C\right).$$

Hence, the $100\big(1-\alpha/(2n)\big)\%$-quantile of the standard normal distribution $N(0,1)$ may serve as an upper estimate of the critical value at the significance level α for the problem (7) applying the test statistic (9). The approximate critical values obtained in this way are good enough for small samples (for small values of n), but they are too conservative for n large.

Therefore, for n large, the asymptotic behavior of the studied test statistic (9) is of interest. It can be proved, applying the law of iterated logarithm, that

$$\max_{0\le k\le n-1}\left\{\left|\frac{1}{\sqrt{n-k}}\sum\nolimits_{i=k+1}^{n} X_i\right|\right\} \longrightarrow \infty \quad \text{almost surely as} \quad n\to\infty.$$

It follows that the limit distribution of (9) does not exist and that critical values increase to infinity as $n\to\infty$. The problem is caused by the behavior of the sequence $\big\{(n-k)^{-1/2}\sum_{i=k+1}^{n} X_i, i=1,\dots,n-1\big\}$ near to its end. Here the averages, whose departures from zero are studied, are calculated only for "a small" number of observations and it can happen, with a large probability, that at least one of them attains a rather large value.

Therefore, some authors suggest to use, instead of the statistic (9), the *trimmed maximum-type* test statistic

$$\max_{0\le k\le \lfloor(1-\beta)n\rfloor}\left\{\left|\frac{1}{\sqrt{n-k}}\sum\nolimits_{i=k+1}^{n} X_i\right|\right\}, \tag{10}$$

where β is a small positive constant less than one and $\lfloor x\rfloor$ denotes the integer part of x. The advantage of the statistic (10) is that it is bounded in probability. The *trimming off* a $100\beta\%$ portion of the sample (upper time points) means that one assumes that the change did not occur during this time period. Notice that, typically, $\beta\in[0.01,0.1]$.

For large n, the approximate critical values can be calculated from the asymptotic behavior of the probabilities under H, because we have for all $x\in\mathcal{R}^1$:

$$P\left(\max_{0\le k\le n-1}\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n} X_i\right|\right\} > \frac{x+b_n}{a_n}\right) \approx 1-\exp\left\{-e^{-x}\right\}, \tag{11}$$

where

$$a_n=\sqrt{2\log\,\log\,n} \quad \text{and} \quad b_n = 2\log\,\log\,n+\frac{1}{2}\log\log\,\log\,n-\frac{1}{2}\log\,\pi,$$

and

$$P\left(\max_{0\le k\le \lfloor(1-\beta)n\rfloor}\left\{\left|\frac{1}{\sqrt{n-k}}\sum_{i=k+1}^{n} X_i\right|\right\} > x\right) \approx 2\big(1-\Phi(x)\big)+x\phi(x)\log\frac{1}{\beta}\cdot \tag{12}$$

9.1.1. Continuation of Example from Section 7.

Suppose that all $n=250$ observations are available in time we try to decide whether a change in mean occurred. The test statistic (9) attains the value 5.353, see Figure 10. Using the Bonferroni inequality we obtain the upper estimate of the corresponding p-value being $2\cdot 10^{-5}$. If we use the asymptotic distribution of the test statistic the estimate of p-value by (11) is 10^{-3}, while it is $3\cdot 10^{-6}$ by applying (12) with $\beta=0.1$. Hence, the null hypothesis H_0 of no change is rejected at the significance level $\alpha=0.001$.

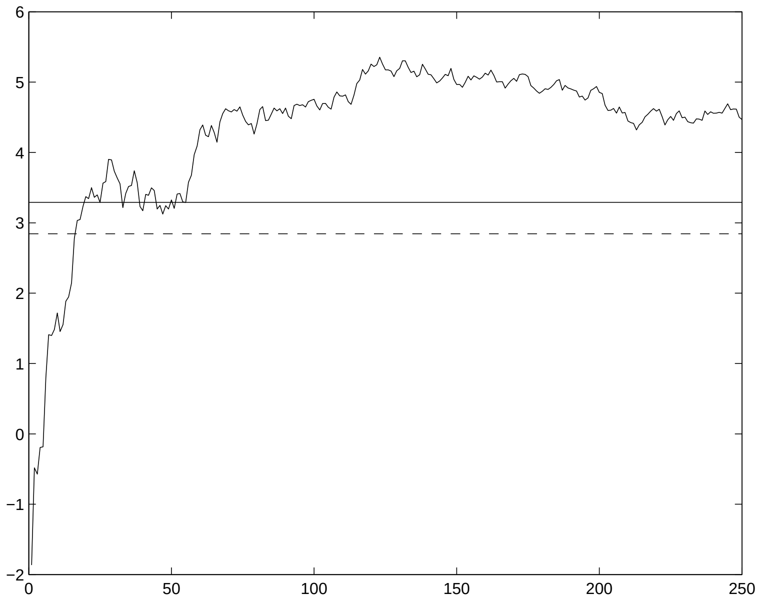

Figure 10. Normalized standardized cumulative sums of data from Section 7. taken in reverse time order with 5% critical values computed according to (11) (solid line) and to (12) with $\beta = 0.1$ (dashed line).

9.2. Detecting a change in mean and/or variance

We have already pointed out that a change in mean is often accompanied by a change in variance. Moreover, both a change in mean and in variance may indicate a trouble. Using a standardization we may suppose that at the beginning the mean $\mu_0 = 0$ and variance $\sigma_0^2 = 1$. The decision whether a change in mean and/or variance of normally distributed random variables occurred may be base on testing the null hypothesis H against the alternative A:

$$\begin{aligned} &H : X_1 \ldots, X_n \sim N(0,\, 1) \\ &A : \exists\, k \,\in \{1, \ldots, n-2\} \text{ such that} \\ &\quad X_1, \ldots, X_k \quad \sim N(0,\, 1), \\ &\quad X_{k+1}, \ldots, X_n \sim N(a, \sigma), \end{aligned} \tag{13}$$

where $a \neq 0$ and/or $\sigma^2 \neq 1$.

The test statistics derived by likelihood ratio approach have the form

$$\max_{1 \le k \le n-2} \widetilde{Z}_k^2 \quad \text{and} \quad \max_{1 \le k \le \lfloor (1-\beta)\, n \rfloor} \widetilde{Z}_k^2, \tag{14}$$

where

$$\widetilde{Z}_k^2 = (n-k) \log \left(\frac{1}{n-k} \sum_{i=k+1}^{n} \left(X_i - \overline{X_k}^{\,o} \right)^2 \right) \,-\, (n-k) \,+\, \sum_{i=k+1}^{n} X_i^2 \tag{15}$$

and

$$\overline{X_k}^{\,o} = \frac{1}{n-k} \sum_{i=k+1}^{n} X_i \, .$$

For n large, the approximate critical values may be computed from the following approximations, i.e.,

$$P\left(\max_{1\le k\le n-2}\widetilde{Z}_k^2>\Big(\frac{x+b_{n,2}}{a_n}\Big)^2\right)\approx 1-\exp\big\{-e^{-x}\big\},\quad x\in\mathcal{R}^1, \tag{16}$$

where

$$a_n=\sqrt{2\log\log\, n}\quad\text{and}\quad b_{n,2}=2\log\log\, n+\log\log\log n,$$

and

$$P\left(\max_{1\le k\le\lfloor(1-\beta)n\rfloor}\widetilde{Z}_k^2>x^2\right)\approx e^{-x^2/2}+\frac{1}{2}e^{-x^2/2}x^2\log\frac{1}{\beta}\cdot \tag{17}$$

9.3. Appearance of gradual linear trend

In practice it can happen that a change in mean does not appear suddenly in a form of a skip (*abrupt change*) but rather it is almost unobservable after the change point and becomes more and more apparent later. Such a change is called *gradual change*. The most simple situation occurs if after the change point the mean of the process increases (or decreases) linearly. In the scope of hypotheses testing we may decide whether such a change occurred by testing the null hypothesis H against the alternative A:

$$\begin{aligned}
&H: X_i=e_i, && i=1,\dots,n,\\
&A: \exists\, k\in\{1,\dots,n-1\}\ \text{ such that}\\
&\qquad X_i=e_i, && i=1,\dots,k,\\
&\qquad X_i=b\cdot\frac{i-k}{n}+e_i, && i=k+1,\dots,n,
\end{aligned} \tag{18}$$

where $b\neq 0$ and $\{e_i\}$ are iid random errors with $\mathsf{E}e_i=0$, $\mathsf{E}e_i^2=1$ and $\mathsf{E}|e_i|^\delta<\infty$ for some $\delta>2$.

The maximum-type statistics have the form

$$\max_{1\le k<n}\left\{\frac{|\,\widehat{b}_k|}{\sqrt{var\,\widehat{b}_k}}\right\}\quad\text{and}\quad\max_{1\le k\le\lfloor(1-\beta)n\rfloor}\left\{\frac{|\,\widehat{b}_k|}{\sqrt{var\,\widehat{b}_k}}\right\}, \tag{19}$$

where $\widehat{b}_k$ is the least squares estimator of b under A supposing that the change occurred at the moment k. Notice that

$$\frac{|\,\widehat{b}_k|}{\sqrt{var\,\widehat{b}_k}}=\frac{\left|\frac{1}{\sigma}\frac{1}{\sqrt{n}}\sum_{i=k+1}^{n}X_i\frac{i-k}{n}\right|}{\sqrt{\frac{(n-k)(n-k+1)(n-k+1/2)}{3n^2}}}. \tag{20}$$

For n large, critical values can be attained using the approximation

$$P\left(\max_{1\le k\le n-1}\left\{\frac{|\,\widehat{b}_k|}{\sqrt{var\,\widehat{b}_k}}\right\}>\frac{x+b_{n,3}}{a_n}\right)\approx 1-\exp\big\{-e^{-x}\big\},\quad x\in\mathcal{R}^1, \tag{21}$$

where

$$a_n=\sqrt{2\ \log\ \log\ n}\quad\text{and}\quad b_{n,3}=2\ \log\ \log\ n+\log\frac{\sqrt{3}}{4\pi},$$

and

$$P\left(\max_{1\le k\le \lfloor (1-\beta)n\rfloor}\left\{\frac{|\widehat{b}_k|}{\sqrt{var\,\widehat{b}_k}}\right\}>x\right)\approx 2\big(1-\Phi(x)\big)+\sqrt{\frac{3}{2\pi}}\,\phi(x)\log\frac{1}{\beta} \qquad (22)$$

For more details see Jarušková (1998).

9.4. Abrupt appearance of linear trend

It can also happen that the change may be of an abrupt type but after the change point the mean does not remain constant but increases or decreases linearly. In this case we test the null hypothesis H against the alternative A:

$$\begin{aligned}
&H : X_i = e_i, && i=1,\dots,n, && (23)\\
&A : \exists\, k \in \{1,\dots,n-2\} \quad \text{such that} && &&\\
&\quad X_i = e_i, && i=1,\dots,k, &&\\
&\quad X_i = a + b\cdot\frac{i}{n} + e_i, && i=k+1,\dots,n, &&
\end{aligned}$$

where $a \neq 0$ and/or $b \neq 0$ where $\{e_i\}$ are the same as in the preceding section.

The test statistics have the form

$$\max_{1\le k\le n-2}\chi_k^2 \quad \text{and} \quad \max_{1\le k\le \lfloor (1-\beta)n\rfloor}\chi_k^2,$$

where

$$\chi_k^2 \;=\; \frac{\left(\sum_{i=k+1}^{n} X_i\right)^2}{n-k} + \frac{\left(\sum_{i=k+1}^{n}\left(\frac{i}{n}-\frac{n-k+1}{2n}\right)X_i\right)^2}{\left(\sum_{i=k+1}^{n}\left(\frac{i}{n}-\frac{n-k+1}{2n}\right)\right)^2}. \qquad (24)$$

For n large, the approximate critical values may be computed from the following approximations, see Albin &Jarušková (2003),

$$P\left(\max_{1\le k\le n-2}\chi_k^2 > \left(\frac{x+b_{n,4}}{a_n}\right)^2\right)\approx 1-\exp\left\{-e^{-x}\right\},\quad x\in\mathcal{R}^1, \qquad (25)$$

where

$$a_n=\sqrt{2\log\log\,n}\quad \text{and}\quad b_{n,4}=2\log\log\,n+\log\log\log n+\log 2,$$

and

$$P\left(\max_{1\le k\le \lfloor (1-\beta)n\rfloor}\chi_k^2 > x\right)\approx e^{-x^2/2}+e^{-x^2/2}x^2\log\frac{1}{\beta}\,. \qquad (26)$$

10. Extremes of limit random processes

It is interesting to realize that even if all the statistics $\left\{(\sum_{i=k+1}^{n} X_i)/\sqrt{n-k}\right\}$ in Subsection 9.1. as well as all the statistics $\{|\widehat{b}_k|/\sqrt{var\,\widehat{b}_k}\}$ defined by (20) in Subsection 9.3. are distributed according to the standard normal distribution, the distributions of the maximum of their absolute values are different. Similarly, all the statistics $\{\widetilde{Z}_k^2\}$ defined by (14) as well as the statistics $\{\chi_k^2\}$ defined by (24) are distributed according to a χ^2

distribution with 2 degrees of freedom, but the exceedence probability of a high level by maximum of variables $\{\chi_k^2\}$ is twice so large as for the variables $\{\widetilde{Z}_k^2\}$.

This can be explained by different type of dependence (different correlation structure) among the variables in the considered sequence. If n is large then the limit distribution of the considered maximum-type test statistic is given by the distribution of the maximum of the corresponding limit process. In the problem (8) the studied limit process is the standardized Wiener process $\{W(t)/\sqrt{t}, 0 < t \leq 1\}$ while in the problem (18) one deals with a maximum of a different Gaussian process $\{\int_0^t (t-s)\, dW(s)/\sqrt{t^3/3 - t^4/4},\, 0 < t \leq 1\}$. The basic difference between these two processes is that the first one has non-differentiable paths while the second one has differentiable paths. The behavior of extremes of differentiable and non-differentiable Gaussian processes was treated in the book Leadbetter et al. (1983).

The behavior of a maximum of the statistics (14) as well as the behavior of a maximum of (24) is given by a maximum of a χ^2 process. A χ^2 process is a process that expresses the square distance of a multivariate (here two-variate) Gaussian process from the origin. The coordinates of the Gaussian process that appears in the problem (13) are independent while they are dependent in (23). The extremes of χ^2 processes were studied by Albin (1990).

11. Concept of research field development and its place in educational system

Statistical concepts and methods are not only useful but indeed often indispensable in understanding the world around us. They provide ways of gaining new insights into the behavior of many phenomena that an engineer encounters in his/her field of specialization in engineering and science.

The discipline of statistics teaches us how to make intelligent judgments and informal decision about a problem where data exhibits random fluctuation. The statistics may be applied in the research when the quality of material for a newly designed product is studied. The stochastic models for lifetimes of product components help us to access the reliability of a product. The statistical analysis of public survey responses shows customer satisfaction with a product or service. However, a specialized branch of mathematical statistics called statistical process control was developed to be applied directly in production processes. The aim of statistical process control is to separate natural random variation in a production process from variation attributable to some assignable causes as contaminated material, incorrect machine setting, unusual tool wear, and the like. Applying statistical process control helps to keep a product quality high.

Quality characteristics of manufactured products have received much attention from design engineers and production personnel as well as those concerned with financial management. An article of faith over the years was that high quality levels and economic well-being were incompatible goals. Recently, however, it has become increasingly apparent that raising quality levels can lead to decreased costs, a greater degree of consumer satisfaction, and thus increased profitability. If the market for the product is competitive, improved quality and reliability can generate very strong competitive advantages. We have seen the results of this in the way that many products, particularly Japanese cars, machine tools, earthmoving equipment, electronic components and consumer electronic products have won dominant positions in world markets in the last 30 to 40 years. Their domination has been largely the result of the teaching of the late W. Edwards Deming, who taught the fundamental connections between quality, productivity and competitiveness. Today this message is well understood by nearly all engineering companies that face the new competition, and those that do not understand lose position or fail.

What is the situation in the Czech Republic? Many factories and production plants try to reach the european standard quality and get the certificate of quality according to the series of standards ISO 900x. This has resulted in renewed emphasis on statistical techniques for designing quality into products and for identifying quality problems at various stages of production and distribution. Doc. Maroš from VŠB Ostrava and doc. Hutyra, the past manager of the company Sezam - the successive organization of Tesla Rožnov, told us a story how the managers of the USA financial group TPG strongly recommended to use control charts for quality characteristics of semiconductor circuits after they bought the division (now ON semiconductor Czech Republic). The extended application of control charts was not only a formal decision but really yielded better quality and decreased number of false products.

What is the role of a statistician in this process? The statistical process control is not a new discipline. As it has already been mentioned, it has been 80 years since Shewhart published his ideas of control charts. The original Shewhart charts were designed to control a mean and variation of a chosen quality characteristic. Nowadays, the measurement techniques enable to measure a large number of characteristics that may be interdependent in many different ways. One problem may cause changes in several characteristics or a change in one characteristic may cause a later change in some others. Moreover, the change need not occur in mean or variation only, but many other types of changes in the stochastic behavior of the quality characteristics may appear. New problems call for new solutions. Unfortunately, the new methods are usually effective if they deal with a problem for which they were developed, but they are quite inefficient when a different type of problem occured. This means that a statistician should cooperate closely with an engineer from the application field who understands the problem from an engineering point of view. On the other hand, the statistician should try to explain to the engineers how the statistical methods work and what they can expect from them.

Originally, the methods for statistical process control were suggested to be applied on line with the objective to detect a problem as soon as possible after it appeared. The designers of control charts supposed that the control charts would be built into the production process or, in the other words, that they will be a part of computer codes that control the production. However, in practice it often happens that the control charts are examined after a whole series of products was produced (e.g. after the end of a factory shift), it means off-line. In this case one is not interested in detecting a problem as soon as possible but rather in making a reliable decision whether a problem occured or not. In the recent time several papers appeared in scientific journals that study the old procedures for on-line approach in the off-line approach context, see e.g. Gut & Steinebach (2003). Of course, the statisticians developed also various methods aimed to be used off-line. With the help of these methods it is possible to detect abrupt change(s) not only in the mean and variance but also for instance in correlation structure or regression dependence. The author of this lecture published several papers that treat the problem of appearance of gradual changes, see Jarušková (1998, 1999, 2002). Unlike the abrupt changes the gradual changes are almost unobservable in the beginning but later become more and more apparent. In these papers she proposed several methods for detection of gradual changes and studied their properties.

Her research is a part of the effort of statisticians to prepare a set of efficient methods for detecting changes in a series of observations. The first papers treating such problems appeared in the seventies, see Hawkins (1977). In 1984 Yao & Davis (1984) obtained the basic result about the behavior of decision rules when they are applied asymptotically, i.e., for a long sequence of observations. In the last 25 years many new theoretical results were obtained but many problems remain unsolved. Unfortunately, it seems that some of these problems will never be solved analytically.

The statistical process control is unfortunately not a part of the curriculum at the Faculty of Civil Engineering. However, during their studies undergraduate as well as graduate students get a basic knowledge how to treat engineering problems where a chance plays a role. Surely, the idea of control charts and decision rules for detecting a change in a production process would not be difficult for them to understand. It is clear that it is not necessary for an engineer to cope with all the mathematical details, rather it is more important for both engineers and mathematicians to be able to communicate and collaborate. Moreover, there exists many computer programs where the basic on – line methods od statistical process control are implemented. Among others let us mention the Matlab statistical toolbox, see Jarušková (1997), which is used in teaching mathematics at the department of mathematics at our faculty. If the model is properly chosen, many problems may be also solved by Monte Carlo methods. Monte Carlo methods are now included into the curriculum of mathematical statistics for PhD students and students of computer science. The students apply for instance Monte Carlo methods to estimate the ARL function for the Shewhart procedures and then they compare it with the analytic result. This way the author of this lecture is trying to raise interest in the statistical quality control methods.

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Jméno autorky **Doc. RNDr. Daniela Jarušková, CSc.**

Vzdělání

1969 - 1974	studium na MFF UK
1975	obhajoba rigorózní práce a získání titulu RNDr.
1974 - 1977	interní aspirantura na MFF UK pod vedním Prof. RNDr. Petra Mandla, DrSc.
1978	obhajoba kandidátské práce a získání titulu CSc.
1990	na základě přednesené přednášky před vědeckou radou FSv ČVUT jmenována docentkou v oboru matematická statistika
2000	obhajoba habilitační práce před vědeckou radou FJFI ČVUT v oboru aplikovaná matematika

Zaměstnání

1977 - 1978	vědecká pracovnice výzkumného ústavu IMADOS
1978 - 1990	odborná asistentka FSv ČVUT
1990 - doposud	docentka FSv ČVUT

Odborné aktivity

Autorka je odbornicí v oboru matematická statistika, pravděpodobnost a stochastická analýza. Během vědecké přípravy v letech 1974–1977 na MFF UK se zabývala problémy náhodných procesů, speciálně stochastickými diferenčními a diferenciálními rovnicemi.

Po nástupu na FSv ČVUT se autorka věnovala analýze časových řad, a to jak v časové, tak i ve frekvenční doméně. Ke studiu časových řad ve frekvenční doméně byla motivována studiem vibrací měřených v různých částech stavební konstrukce. Metody ke zpracování časových řad v časové doméně pak našly aplikaci v hydrologii a klimatologii.

V posledních 15 letech se autorka zabývá problémem bodu změny. Problémy bodu změny tvoří rozsáhlou část matematické statistiky, která zkoumá, zda sledovaná časová řada může být považována za stacionární nebo zda se stochastické chování řady během pozorování mění. Je zřejmé, že použití metod pro odhalování změn je velmi rozsáhlé. Autorka aplikovala tyto metody na problémy klimatologie a hydrologie (globální oteplování), na problémy statistického řízení kvality (Shewhartovy a CUSUM diagramy) i na problémy medicíny (chování mytochondrií). V teoretické oblasti vede analýza bodu změny často k velmi složitým matematickým problémům. Řešení některých z těchto problémů autorka publikovala ve významných zahraničních časopisech. Při řešení autorka spolupracuje s Prof. Marií Huškovou, DrSc. a Prof. Jaromírem Antochem, CSc. v rámci grantu 201-03-0945, 2003 – 2005.

Autorka se však též velmi hluboce zajímá o praktické aplikace matematické statistiky při studiu spolehlivosti konstrukcí. V rámci záměru MSM 210000001 spolupracuje velmi úzce s pracovníky katedry stavební mechaniky Prof. Ing. Jiřím Šejnohou, DrSc. a Ing. Marií Kalouskovou, CSc. při odhadování spolehlivosti konstrukcí a jejich prvků, dále spolupracuje s pracovníky katedry ocelových a dřevěných konstrukcí Doc. Ing. Petrem Kuklíkem, CSc. a Ing. Annou Kuklíkovou na metodách pro zjišťování pevnosti prvků dřevěných konstrukcí.

Pracovní pobyty a přednášky

Autorka je často zvána k přednesení přednášky na mezinárodní konferenci do zahraničí – z poslední doby jmenujme Berlín (Německo 1999), Portland (USA 2001), Neuchâtel (Švýcarsko 2002), Kijev (Ukrajina 2002). Byla zvanou řečnicí i na mezinárodních konferencích konaných v České republice (Praha 1998, Brno 1999) a byla i hlavní zvanou řečnicí na české statistické konferenci ROBUST'04, která se konala v tomto roce. Kromě toho je autorka zvána i k přednáškám na zahraničních univerzitách – ETH Zürich (Švýcarsko 1994), Univerzita v Neapoli (Itálie 1997, 2002), Univerzita v Barceloně (Španělsko 1998), Univerzita v Toulouse (Francie 1999), Water Research Institute Burlington (Kanada 2001), Univerzita v Cagliari (Itálie 2001).

V roce 2000 byla půl roku hostujícím profesorem na Michigan State University v East Lansingu, Michigan, USA.

Autorka absolvovala dva odborné studijní pobyty – dvouměsíční pobyt v roce 1984 na Univerzitě v Patrasu (Řecko) a čtyřměsíční pobyt v letech 1994–1995 na ETH Zürich (Švýcarsko).

Spolupráce s jinými pracovišti

V letech 1992–1995 spolupracovala s pracovníky Českého hydrometeorologického ústavu. V rámci spolupráce vzniklo několik výzkumných zpráv a počítačových programů. Krátkodobě autorka spolupracovala s pracovníky Výzkumného ústavu zemědělského a výzkumnými pracovníky ŠKODA – Plzeň a Energoprojekt.

Dlouhodobě autorka spolupracuje především s pracovníky katedry matematické statistiky MFF UK a pracovníky Université Bordeaux na problémech spojených s odhadováním spolehlivosti. V rámci spolupráce navštívila 5x univerzitu v Bordeaux. Tato spolupráce je částečně podporována česko-francouzským projektem Barrande.

Autorka je členkou mezinárodní Bernoulliho společnosti (součást Mezinárodního statistického institutu), Jednoty matematiků a fyziků a České statistické společnosti, kde působila dva roky ve výboru.

Publikační a recenzní činnost

Autorka publikovala 10 článků v zahraničních časopisech a dvě rozsáhlé kapitoly 125 stran v mezinárodní monografii. Dále publikovala 5 článků v českých odborných časopisech. Ohlas v odborné veřejnosti ilustruje to, že její práce byly 40 krát citovány v zahraničních i našich časopisech. Kromě toho publikovala 6 článků ve sbornících mezinárodní konferencí a 13 článků ve sbornících českých konferencí. Je autorkou 3 skript pro studenty FSv ČVUT. Tato skripta byla opakovaně citována nejen českými, ale i slovenskými autory. Recenzovala asi 16 odborných článků, především pro mezinárodní časopisy Journal of Statistical Planning and Inference, Statistics and Probability Letters, Journal of Climatology a Kybernetika. Dále píše odborné reference pro časopis Zentralblatt (asi 50).

Pedagogické aktivity

Autorka přednáší i cvičí na FSv ČVUT předmět matematická statistika. Dále přednáší předmět matematická statistika I a matematická statistika II pro doktorandy. V letech 1990–2000 byla členkou komise pro obhajoby diplomových prací a pro státní závěrečné zkoušky na oboru konstrukce a doprava. Od roku 1994 je členkou komise pro státní závěrečné zkoušky a obhajoby diplomových prací na oboru matematická statistika MFF UK. V roce 2000 přednášela půl roku na Michigan State University.

Vědecká výchova, vedení vědeckého týmu

Autorka pomáhala při přípravě doktorské práce velkému množství doktorandů z FSv ČVUT. Z některými z nich publikovala též články v odborných časopisech (Ing. M. Bořík, Ing. E. Novotná). V současné době se chystá k odevzdání doktorské práce Ing. Radim Kraut z Technické univerzity v Ostravě. Autorka (jako vedlejší školitelka) vedla jeho práci od samého počátku. Ze vzájemné spolupráce vznikly tři publikace. Dále je školitelkou dvou doktorandů na FSv ČVUT – Mgr. Moniky Rencové a Ing. Jana Bryscejna a jednoho doktoranda na MFF UK Mgr. Martina Haneka.

Autorka vedla také dvě diplomové práce na MFF UK. Obě práce byly ohodnoceny výborně a autorka druhé z nich současně studuje doktorandské studium v USA. Pod vedením Doc. Jaruškové též každoročně pracuje jeden pomvěd. Pomvěd Aleš Menšík spolupracoval na publikování odborného článku a zpracovával data v rámci spolupráce s Českým hydrometeorologickým ústavem. Současný pomvěd Jan Záruba připravil počítačové programy pro prezentaci doktorandky Moniky Rencové na zahraniční konferenci.

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Ocenění

V roce 2001 získala na návrh děkana FSv ČVUT zvláštní ocenění rektora ČVUT.