České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Ing. Petr Cintula, Ph.D.

Matematická fuzzy logika od teorie k aplikacím

Mathematical Fuzzy Logic From Theory to Applications

Summary

Formal logic as the science of different forms of correct reasoning began with the philosophy of ancient Greece, became a fully fledged mathematical discipline (known as mathematical logic) in the 19th and 20th centuries, and currently plays an essential role especially in modern (theoretical) computer science.

The goal of the lecture is to present an introduction to Mathematical *Fuzzy* Logic, a subdiscipline of Mathematical Logic studying certain family of formal logical systems whose algebraic semantics involves the notion of *truth degrees*.

We start by a description of three different origins/motivations of this research area. Then we present the Łukasiewicz logic, an important example of a particular fuzzy logic. We continue by presenting an abstract logical framework created to cope with a whole extensive family of fuzzy logics. We conclude by sketching a theory combining fuzzy logic and the theory of probability.

Souhrn

Formální logika jako věda o různých formách správného usuzování, jejíž počátky jsou spjaty s filozofií starověkého Řecka, se v 19. a 20. století stala plně rozvinutou disciplínou matematiky (matematickou logikou) a v současné době hraje významnou roli zejména v moderní (teoretické) informatice.

Cílem přednášky je prezentovat úvod do matematické *fuzzy* logiky, disciplíny matematické logiky studující třídu formálních logických systémů, jejichž algebraická sémantika pracuje s pojmem *stupňů pravdivosti*.

Přednášku zahájíme popisem tří různých východisek/motivací této oblasti výzkumu. Poté představíme Łukasiewiczovu logiku, důležitý příklad konkrétní fuzzy logiky. Dále budeme pokračovat prezentací abstraktní logické teorie vytvořené pro zachycení celé rozvětvené rodiny fuzzy logik. Na závěr nastíníme teorii kombinující fuzzy logiku a teorii pravděpodobnosti.

Klíčová slova

Matematická logika, matematická fuzzy logika, stupně pravdivosti, vágnost, Łukasiewiczova logika, MV-algebry, abstraktní algebraická logika, relace důsledku, slabě implikativní logiky, semilineární logiky, teorie pravděpodobnosti

Keywords

Mathematical logic, mathematical fuzzy logic, truth degrees, vagueness, Łukasiewicz logic, MV-algebras, abstract algebraic logic, consequence relation, weakly implicative logics, semilinear logics, probability theory

Contents

1 Introduction

Formal logic as the science of different forms of correct reasoning began with the philosophy of ancient Greece, became a fully fledged mathematical discipline (known as mathematical logic) in the 19th and 20th centuries, and currently plays an essential role especially in modern (theoretical) computer science.

The goal of the lecture is to present an introduction to Mathematical *Fuzzy* Logic, a subdiscipline of Mathematical Logic studying certain family of formal logical systems whose algebraic semantics involve some notion of truth degrees.

We start by a short description of three different origins/motivations of this research area. Then we present the Łukasiewicz logic, an important example of a particular fuzzy logic. We continue by presenting an abstract logical framework created to cope with a whole extensive family of fuzzy logics. We conclude by sketching a theory combining fuzzy logic and the theory of probability.

2 Mathematical Fuzzy Logic: motivation and history

The central rôle of truth degrees in Mathematical Fuzzy Logic (MFL) stems from three distinct historical origins of the discipline:

Philosophical motivations Any scientific theory is, at least initially, driven by some kind of external motivation, i.e. some independent reality one would like to understand and model by means of the theory. MFL is motivated by the need to model correct reasoning in some particular contexts where more standard systems, such as classical logic, might be considered inappropriate. Namely, these motivating contexts are those where the involved propositions suffer from a lack of precision, typically because they contain some vague predicate, i.e. a property lacking clear boundaries.

Vague predicates (such as 'tall', 'intelligent', 'poor', 'young', 'beautiful', or 'simple') are omnipresent in natural language and reasoning and, thus, dealing with them is also unavoidable in linguistics. They constitute an important logical problem as clearly seen when confronting *sorites* paradoxes, where a sufficient number of applications of a legitimate deduction rule (*modus ponens*) leads from (apparently?) true premises, to a clearly false conclusion: (1) one grain of wheat does not make a heap, (2) a group of grains of wheat does not become a heap just by adding one more grain, therefore: (3) one million grains of wheat does not make a heap.

One possible way to tackle this problem is the degree-based approach related to logical systems studied by MFL (for survey of other logical approaches see e.g. [32]). In this proposal one assumes that truth comes in degrees which, in the case of the *sorites* series, vary from the absolute truth of 'one grain of wheat does not make a heap' to the absolute falsity of 'one million grains of wheat does not make a heap', through the intermediate decreasing truth degrees of 'n grains of wheat do not make a heap'.

Fuzzy Set Theory In 1965 Lotfi Zadeh proposed fuzzy sets as a new mathematical paradigm for dealing with imprecision and gradual change in engineering applications [39]. Their conceptual simplicity (a fuzzy set is nothing more than a classical set endowed with a $[0, 1]$-valued function which represents the degree to which an element belongs to the fuzzy set) provided the basis for a substantial new research area and applications such as a very popular engineering toolbox used successfully in many technological applications, in particular, in so-called *fuzzy control*.

This field is referred to as *fuzzy logic*, although its mathematical machinery and the concepts investigated are largely unrelated to those typically used and studied in (Mathematical) Logic. Nevertheless, there have been some early attempts to present fuzzy logic in the sense of Zadeh as a useful tool for dealing with vagueness paradoxes (see e.g. [25]).

Many-valued logics The 20th century witnessed a proliferation of logical systems whose intended algebraic semantics, in contrast to classical logic, have more than two truth values (for a historical account see e.g. [21]). Prominent examples are 3-valued systems like Kleene's logic of indeterminacy and Priest's logic of paradox, 4-valued systems like Dunn–Belnap's logic, n-valued systems of Łukasiewicz and Post, and even infinitely valued logics of Łukasiewicz logic [34] or Gödel–Dummett logic [13]. These systems were inspired by a variety of motivations, only occasionally related to the aforementioned vagueness problems.

More recently, Algebraic Logic has developed a paradigm in which most systems of non-classical logics can be seen as many-valued logics, because they are given a semantics in terms of algebras with more than two truth values. From this point of view, many-valued logics encompass wide well-studied families of logical systems such as relevance logics, intuitionistic and superintuitionistic logics and substructural logics in general (see e.g. [22]).

Mathematical Fuzzy Logic was born at the crossroads of these three areas. At the beginning of the nineties of last century, a small group of researchers (including among others Esteva, Godo, Gottwald, Hájek, Höhle, and Novák), persuaded that fuzzy set theory could be a useful paradigm for dealing with logical problems related to vagueness, began investigations dedicated to providing solid logical foundations for such a discipline.

In other words, they started developing logical systems in the tradition of Mathematical Logic that would have the $[0,1]$-valued operations used in fuzzy set theory as their intended semantics. In the course of this development, they realised that some of these logical systems were already known such as Łukasiewicz and Gödel–Dummett infinitely valued logics. Both systems turned out to be strongly related to fuzzy sets because they are $[0,1]$-valued and the truth functions interpreting their logical connectives are, in fact, of the same kind (t-norms, t-conorms, negations) as those used to compute the combination (resp. intersection, union, complement) of fuzzy sets.

These pioneering efforts produced a number of important papers and even some monographs (especially [27], but also [26, 37]) and established fuzzy logics as a respectable family in the broad landscape of non-classical logics studied by Mathematical Logic. In order to distinguish the study of these logics from the works on fuzzy set theory misleadingly labeled as *fuzzy logic*, the term *Mathematical Fuzzy Logic* was coined.

Moreover, being a subdiscipline of Mathematical Logic, MFL has acquired the typical core agenda of this field and is studied by many mathematically-minded researchers regardless of its original motivations. The last years have seen the blossoming of MFL with a plethora of works on propositional, modal, predicate (first and higher order) logics, their semantics (algebraic, relational, game-theoretic), proof theory, model theory, complexity and (un)decidability issues, etc. These works culminated in a two-volume Handbook of Mathematical Fuzzy Logic [8] which provides an up-to-date systematic presentation of the best-developed areas of MFL.

3 An example: Łukasiewicz infinite-valued logic

Łukasiewicz infinite-valued logic Ł is nowadays reputedly the most developed fuzzy logic. It was introduced in [34] and was later studied in numerous papers, two extensive monographs [5, 36] and the handbook chapter [33].

Its formulas are built from a (countable) set of propositional variables using two basic connectives: a binary connective $\to$ and a truth constant $\overline{0}$. The real unit interval serves as the *standard* (or intended) set of truth degrees, and the basic connectives are interpreted using function $\min\{1, 1-x+y\}$ and constant 0.

In Łukasiewicz logic we also use additional derived connectives (we list them with their standard semantics):

$\overline{1}$	is	$\overline{0} \to \overline{0}$	1
$\neg\varphi$	is	$\varphi \to \overline{0}$	1 - x
$\varphi \vee \psi$	is	$(\varphi \to \psi) \to \psi$	$\max\{x, y\}$
$\varphi \wedge \psi$	is	$\neg(\neg\varphi \vee \neg\psi)$	$\min\{x, y\}$
$\varphi \oplus \psi$	is	$\neg\varphi \to \psi$	$\min\{1, x+y\}$
$\varphi \otimes \psi$	is	$\neg(\neg\varphi \oplus \neg\psi)$	$\max\{0, x+y-1\}$
$\varphi \ominus \psi$	is	$\neg(\varphi \to \psi)$	$\max\{0, x-y\}$
$\varphi \leftrightarrow \psi$	is	$(\varphi \to \psi) \wedge (\psi \to \varphi)$	$\|x-y\|$

Definition 1 (Hilber-Style axiomatic systems [34])**.** *The axiomatic system of Łukasiewicz logic has the* deduction rule *modus ponens (from φ and $\varphi \to \psi$ infer ψ) and the* axioms*:*

(1) $\varphi \to (\psi \to \varphi)$

(2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

(4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

The basic syntactic notions of *theory, proof, provability* are defined as in the classical logic: a formula φ is *provable* from a *theory* (a set of formulas) T ($T \vdash \varphi$ in symbols) is there is a *proof*: a finite sequence of formulas ending with φ, where each element χ is an instance of an axiom, an element of T, or it is preceded by formulas ψ and $\psi \to \chi$ (i.e., we applied the deduction rule of *modus ponens*).

Now we recall the general algebraic semantics for Łukasiewicz logic, the class of MV-algebras. By abuse of language, we use the same symbols to denote logical connectives and the corresponding algebraic operations.

Definition 2 (Algebraic semantics[4])**.** *An* MV*-algebra is a structure* $\boldsymbol{A} = (A, \oplus, \neg, \overline{0})$ *such that the following conditions are satisfied:*

(MV1) $(A, \oplus, \overline{0})$ *is a commutative monoid,*
(MV2) $x \oplus \neg\overline{0} = \neg\overline{0}$,
(MV3) $\neg\neg x = x$,
(MV4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In each MV-algebra, we define the additional connectives: $x \otimes y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$, $x \ominus y = \neg(\neg x \oplus y)$, $x \vee y = (x \ominus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$, and $\overline{1} = \neg\overline{0}$. It can be shown that the reduct $(A, \vee, \wedge, \overline{0}, \overline{1})$ forms a bounded lattice, thus in each MV-algebra we can introduce the lattice order $\leq_{\boldsymbol{A}}$. If $\leq_{\boldsymbol{A}}$ is a linear order we say that the given MV-algebra is linearly ordered (or that it is an MV-chain). An $\boldsymbol{A}$-evaluation e is a mapping from the set of all formulas into $\boldsymbol{A}$ such that: $e(\overline{0}) = 0$ and $e(\varphi \to \psi) = e(\varphi) \to e(\psi) = \neg e(\varphi) \oplus e(\psi)$.

The MV-chain with the real unit interval as a domain, the operation $\oplus$ interpreted as $\min(0, x + y)$, $\neg$ interpreted as $1 - x$, and $\overline{0}$ as 0 is called the *standard* MV-*algebra* (or the *standard algebra of Łukasiewicz logic*) and denoted by $[0, 1]_{Ł}$.

Theorem 1 (Completeness theorem [4])**.** *Let T be a theory and φ a formula. Then the following are equivalent:*

- $T \vdash \varphi$.
- $e(\varphi) = 1$ *for each* MV-algebra $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-evaluation* e *such that* $e(\psi) = 1$ *for each* $\psi \in T$.
- $e(\varphi) = 1$ *for each* MV-chain $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-evaluation* e *such that* $e(\psi) = 1$ *for each* $\psi \in T$.

If the theory T is finite *we can also add the condition:*

- $e(\varphi) = 1$ *for each* $[0, 1]_{Ł}$*-evaluation* e *such that* $e(\psi) = 1$ *for each* $\psi \in T$.

It is worth mentioning the problem of proving a formula from a finite theory is NP-complete; perhaps surprising result, taking in account the increase of complexity of semantic of Ł relative to the classical logic. The modern development of Łukasiewicz logic mainly concentrates on geometric properties of its (standard) semantics and tight relation of MV-algebras and Abelian lattice ordered groups.

4 General theory of fuzzy logics

As mentioned above the seminal monograph of MFL was Hájek's [27] published in 1998. This book studied three most prominent fuzzy logics (Łukasiewicz, Product, and Gödel–Dummett). Moreover, in order to provide a common ground for these logics, the monograph presented Basic Fuzzy Logic BL, a common base logic that could be axiomatically extended to all three.

This monograph began a process in which an increasing number of researchers have contributed by proposing a growing collection of systems of fuzzy logics obtained by modifying the defining conditions of BL and its three main extensions. For instance, some properties of conjunction were dropped and a new 'basic logic' MTL was proposed in [16], many axiomatic extensions of MTL were studied [14], negation was removed [17], commutativity of conjunction was disregarded [28]. On the other hand, logics with a higher expressive power were introduced by considering additional connectives, e.g. 0–1 projection [1], involutive negation [18], or truth-constants [15]. In accordance with their initial motivations, the proponents of all these systems have always borne in mind an intended (so called *standard*) semantics based on real-valued algebras, and tried to show completeness of their new logics with respect to such algebras. However, in recent works fuzzy logics have begun a process of emancipation from real-valued algebras by considering systems complete with respect to rational, finite or hyperreal linearly ordered algebras, see e.g. [7].

When dealing with this huge variety of fuzzy logics, and in order to avoid a useless repetition of analogous results and proofs, one may want to have some general theory able to cope with all known examples of fuzzy logics and with other new logics that may arise in the near future. In doing so, one certainly needs some intuition about the class of objects one would like to mathematically determine, namely some intuition of what are the minimal properties that should be required for a logic to be considered *fuzzy*.

Papers [2, 6] proposed and defended the thesis that *'fuzzy logics are logics of chains',* i.e., logics whose semantics somehow involves totally ordered set(s) of truth degrees. The reasons for this (so far only informal) definition are both 'pragmatic' (almost all logics studied in the literature as fuzzy logics satisfy this condition and vice-verse) and 'philosophical' (indeed we have argued that fuzzy logics investigate the *degrees of truth,* and it is natural to assume that *degrees,* are *comparable*).

In order give a technical mathematical rendering of this thesis, i.e., to give *mathematical* answer to the question "Which logics are fuzzy?", we first need to answer the question "What is a logic (as a mathematical object)". A possible and relatively simple answer identifies logics with *structural consequence relations*, i.e., special relations between sets of formulas and formulas [38]. This is a natural generalization of the provability relation whose definition we have illustrated for Łukasiewicz logic; therefore an infix notation is often used, i.e., we write $T \vdash_{\mathrm{L}} \varphi$ instead of $\langle T, \varphi \rangle \in \mathrm{L}$.

For our purposes we simplify the context even more:

Definition 3 ([6])**.** *A logic* L *is* weakly implicative *if its language contains a (definable) binary connective* $\to$ *such that:*

$$\begin{aligned}
&\vdash_{\mathrm{L}} \varphi \to \varphi \\
\varphi, \varphi \to \psi &\vdash_{\mathrm{L}} \psi \\
\varphi \to \psi, \psi \to \chi &\vdash_{\mathrm{L}} \varphi \to \chi \\
\varphi \to \psi, \psi \to \varphi &\vdash_{\mathrm{L}} c(\dots, \varphi, \dots) \to c(\dots, \psi, \dots) \text{ for all connectives } c
\end{aligned}$$

To each logic L we can assign a class of its *logical matrices,* denoted as $\mathrm{MOD}^*(\mathrm{L})$, i.e., algebras equipped with a subset of *designated elements* (cf. the class of MV-algebras, where the set F would always be the singleton $\{\overline{1}\}$). For each matrix $\langle \boldsymbol{A}, F\rangle$ we can define the notion of $\boldsymbol{A}$-evaluation analogously to what we have seen the case of Łukasiewicz logic. The resulting abstract completeness theorem is a natural generalization of the completeness theorem of Łukasiewicz logic.

Theorem 2 ([11])**.** *Let* L *be a logic,* T *a theory, and* φ *a formula.*

$$T \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad e(\varphi) \in F \textit{ for each } \langle \boldsymbol{A}, F\rangle \in \mathrm{MOD}^*(\mathrm{L}) \textit{ and each } \boldsymbol{A}\textit{-evaluation } e \textit{ such that } e[T] \subseteq F\,.$$

A crucial property of weakly implicative logic is that their matrices can be naturally *ordered*: taking any weakly implicative logic L and any matrix $\langle \boldsymbol{A}, F\rangle \in \mathrm{MOD}^*(\mathrm{L})$ we can define an order $\leq_{\langle \boldsymbol{A}, F\rangle}$ as:

$$x \leq_{\langle \boldsymbol{A}, F\rangle} y \ \equiv_{\mathrm{df}} \ x \to^{\boldsymbol{A}} y \in F\,.$$

This allows us (analogously as in Ł), for a given weakly implicative logic L, to define the class of its *linearly ordered matrices* denoted as $\mathrm{MOD}^\ell(\mathrm{L})$ and formulate our central definition.

Definition 4 ([6])**.** *A weakly implicative logic* L *is* semilinear *whenever it is complete w.r.t. the class of its* linearly *ordered matrices, i.e.,*

$$T \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad e(\varphi) \in F \textit{ for each } \langle \boldsymbol{A}, F\rangle \in \mathrm{MOD}^\ell(\mathrm{L}) \textit{ and each } \boldsymbol{A}\textit{-evaluation } e \textit{ such that } e[T] \subseteq F\,.$$

The class of weakly implicative *semilinear* logics should be seen as a *formal* approximation of an *intuitive* notion of fuzzy logic (cf. the analogous situation with mathematical approximation of the informal notion of algorithm).[1] It needs to be stressed that while this mathematical

[1] The technical term 'semilinear' is used instead perhaps more natural term 'fuzzy', which however is too heavily charged with many conflicting potential meanings, so more neutral name was chosen.

definition aims to encompass the majority of existing fuzzy logics, we do not expect to capture the whole *intuitive* notion of arbitrary fuzzy logic. Indeed there might still be several other ways in which a logic might be 'a logic of chains' (see e.g. [3]). But still, the notion of semilinear logic provides a good approximation of our informal notion of fuzzy logic from both pragmatic and philosophical perspective and above all it provides a useful mathematical framework for general study of a wide class of non-classical logics. For details see the extensive 100+ pages long handbook chapter [9].

5 Combining fuzzy logic and probability

At this point is should be clear that degrees of truth are conceptually very different from degrees of probability despite sharing the real unit interval as their domain. Compare the sentences 'It will probably rain tomorrow' and 'It rains heavily'. The former expresses our uncertainty/lack of knowledge: tomorrow it will rain or it will not. The latter one is vague; even if we measure exactly how much water is coming down per minute per square centimeter, the problem of vagueness will not go away (recall the sorites paradox described in Section 2).

Other fundamental difference is in assigning truth/probability degrees to composed events: in fuzzy logic the truth degree of formula $\varphi \wedge \psi$ is computed using a fixed function (in this case minimum) from the truth degrees of subformulas φ and ψ. This is not the case in the probability theory: probability of joint event is also determined by the degree of dependency of the subevents. We say that (fuzzy) logic is *extensional* or *truth functional;* whereas probability is not.

Despite these fundamental differences there were numerous attempts to combine (fuzzy) logic and probability. Our approach is based on Hamblin's seminal idea [31] of enriching classical logic by a modal operator $\Box\varphi$, read as 'probably φ', with intended meaning that $\Box\varphi$ is true if the probability of φ is bigger than or equal to a given threshold.

This idea was later elaborated by Fagin, Halpern and many others (see e.g. [30]), who considered uncountably many modalities $\Box_\alpha$ with intended meaning $\Box_\alpha\varphi$ is true if the probability of φ is bigger than or equal to α. Paper [29] presented an idea (later developed in [27]) to replace this heavy machinery of uncountably many modalities by one *fuzzy* modality and say that the *truth degree* of $\Box\varphi$ equals the probability of φ.[2]

[2] Note that this is fundamentally different from saying that *truth degree* of φ equals the probability of φ. Doing this would immediately lead to problems with the truth-functionality mentioned above.

In order to avoid possible complications we use the so-called two-layer syntax with: (i) non-modal formulas of classical logic, (ii) atomic modal formulas obtained by applying the modality operator $\Box$ only to non-modal ones, and (iii) complex modal formulas of Łukasiewicz logic built from the atomic ones. This yields a two-layer syntax where modalities are never nested, can only be applied to formulas of classical logic, and the intended reading of atomic modal formulas $\Box\varphi$ is 'probably φ' (or 'the probability of φ is high').

The intended semantics of our 'Łukasiewicz probability logic over classical logic', denoted FP(Ł), are the so-called *probability Kripke frames/models*, formalizing the idea that non-modal classical formulas should be evaluated in the classical two-valued way, a probability measure is used for interpretation of atomic modal formulas, and modal formulas are then interpreted in Łukasiewicz logic.

Definition 5 ([27]). *A* probability Kripke frame *is a structure* $\mathbf{F} = \langle W, \mu \rangle$*, where* W *is a set of possible worlds, and* μ *is a finitely additive probability measure.*

A Kripke model *over a probability Kripke frame* $\mathbf{F} = \langle W, \mu \rangle$ *is a structure* $\mathbf{M} = \langle \mathbf{F}, \langle e_w \rangle_{w \in W} \rangle$ *where:*

- e_w*s are classical evaluations of non-modal formulas*
- *the set* $\{w \mid e_w(\varphi) = 1\}$ *is in the domain of* μ*, for each non-modal formula* φ*.*

The truth value of non-modal formulas in each world $w \in W$ is interpreted by the corresponding classical evaluation e_w, the truth value of an atomic modal formula $\Box\varphi$ is uniformly defined as

$$||\Box\varphi||_{\mathbf{M}} = \mu(\{w \mid e_w(\varphi) = 1\}),$$

and truth values of complex modal formulas are obtained by means of the operations in $[0,1]_{Ł}$. This gives a semantics for both modal and non-modal formulas of FP(Ł).

Interestingly enough the rather complex semantics we have just defined (which involves all possible finitely additive probability measures) can be axiomatized by a relatively simple axiomatic system:

Definition 6 ([27]). *The logic* FP(Ł) *has the axiomatic system:*

- *axioms of classical propositional logic for non-modal formulas and axioms of Łukasiewicz logic* Ł *for modal ones,*
- modus ponens *rules for both non-modal and modal formulas,*

- *additional axioms:*

  A1 $\Box\varphi \to_{Ł} (\Box(\varphi \to \psi) \to_{Ł} \Box\psi)$

  A2 $\Box\neg\varphi \leftrightarrow_{Ł} \neg_{Ł}\Box\varphi$

  A3 $\Box(\varphi \vee \psi) \leftrightarrow_{Ł} [(\Box\varphi \to_{Ł} \Box(\varphi \wedge \psi)) \to_{Ł} \Box\psi]$

- *a modal rule of necessitation* $\varphi \vdash \Box\varphi$*.*

Theorem 3 (Completeness theorem [27]). *Let* Γ *be a finite set of modal formulas and* δ *a modal formula. Then the following are equivalent:*

- $\Gamma \vdash_{\mathrm{FP(Ł)}} \delta$
- $||\delta||_{\mathbf{M}} = 1$ *for each probability Kripke frame* $\mathbf{F}$ *and each probability Kripke model* $\mathbf{M}$ *over* $\mathbf{F}$ *such that* $||\gamma||_{\mathbf{M}} = 1$ *for each* $\gamma \in \Gamma$*.*

Several works have followed this idea with variations. In [23] Godo, Esteva and Hájek replaced Łukasiewicz logic by a more expressive logic ŁΠ [19] which enabled them to deal with conditional probability. Godo and Marchioni investigated coherent conditional probabilities in [24]. Marchioni also proposed a class of *logics of uncertainty* in [35] with different kinds of measures besides probability (e.g. possibility and necessity measures, see [12]) to quantify the uncertainty of events. In all of these works classical logic has been kept as the underlying logic for non-modal formulas.

Furthermore if one wants to deal with uncertainty and vagueness at once, i.e. with the probability of vague events, as in 'tomorrow it will probably rain *heavily*', the two-layer paradigm can still be useful provided that the underlying classical logic is substituted by a suitable fuzzy logic. This idea has been also investigated in some works, as [20] where finite Łukasiewicz systems $Ł_n$ are taken as the logics of vague events. Recently a paper [10] provided a new general framework for two-layer modal fuzzy logics that encompasses the current state of the art and paves the way for future development.

References

[1] M. Baaz. Infinite-valued Gödel logic with 0-1-projections and relativisations. In P. Hájek, editor, *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics*, volume 6 of *Lecture Notes in Logic*, pages 23–33. Springer-Verlag, Brno, 1996.

[2] L. Běhounek and P. Cintula. Fuzzy logics as the logics of chains. *Fuzzy Sets and Systems*, 157(5):604–610, 2006.

[3] F. Bou, F. Esteva, J. M. Font, À. Gil, L. Godo, A. Torrens, and V. Verdú. Logics preserving degrees of truth from varieties of residuated lattices. *Journal of Logic and Computation*, 19(6):1031–1069, 2009.

[4] C. C. Chang. Algebraic analysis of many-valued logics. *Transactions of the American Mathematical Society*, 88(2):467–490, 1958.

[5] R. Cignoli, I. M. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordrecht, 1999.

[6] P. Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[7] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna, and C. Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies. *Annals of Pure and Applied Logic*, 160(1):53–81, 2009.

[8] P. Cintula, P. Hájek, and C. Noguera, editors. *Handbook of Mathematical Fuzzy Logic (in 2 volumes)*, volume 37, 38 of *Studies in Logic, Mathematical Logic and Foundations*. College Publications, London, 2011.

[9] P. Cintula and C. Noguera. A general framework for mathematical fuzzy logic. In P. Cintula, P. Hájek, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic - Volume 1*, volume 37 of *Studies in Logic, Mathematical Logic and Foundations*, pages 103–207. College Publications, London, 2011.

[10] P. Cintula and C. Noguera. Modal logics of uncertainty with two-layer syntax: A general completeness theorem. In U. Kohlenbach, P. Barceló, and R. J. de Queiroz, editors, *Logic, Language, Information, and Computation - WoLLIC 2014*, volume 8652 of *Lecture Notes in Computer Science*, pages 124–136. Springer, 2014.

[11] J. Czelakowski. *Protoalgebraic Logics*, volume 10 of *Trends in Logic*. Kluwer, Dordrecht, 2001.

[12] D. Dubois and H. Prade. *Possibility Theory*. Plenum Press, New-York, 1988.

[13] M. Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 24(2):97–106, 1959.

[14] F. Esteva, J. Gispert, L. Godo, and F. Montagna. On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica*, 71(2):199–226, 2002.

[15] F. Esteva, J. Gispert, L. Godo, and C. Noguera. Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results. *Fuzzy Sets and Systems*, 158(6):597–618, 2007.

[16] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[17] F. Esteva, L. Godo, P. Hájek, and F. Montagna. Hoops and fuzzy logic. *Journal of Logic and Computation*, 13(4):532–555, 2003.

[18] F. Esteva, L. Godo, P. Hájek, and M. Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[19] F. Esteva, L. Godo, and F. Montagna. The ŁΠ and Ł$\Pi\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[20] T. Flaminio and L. Godo. A logic for reasoning about the probability of fuzzy events. *Fuzzy Sets and Systems*, 158(6):625–638, 2006.

[21] D. M. Gabbay and J. Woods, editors. *The Many Valued and Nonmonotonic Turn in Logic*, volume 8 of *Handbook of the History of Logic*. North-Holland, Amsterdam, 2007.

[22] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[23] L. Godo, F. Esteva, and P. Hájek. Reasoning about probability using fuzzy logic. *Neural Network World*, 10(5):811–823, 2000. Special issue on SOFSEM 2000.

[24] L. Godo and E. Marchioni. Coherent conditional probability in a fuzzy logic setting. *Logic Journal of the Interest Group of Pure and Applied Logic*, 14(3):457–481, 2006.

[25] J. A. Goguen. The logic of inexact concepts. *Synthese*, 19(3–4):325–373, 1969.

[26] S. Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[27] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.

[28] P. Hájek. Fuzzy logics with noncommutative conjunctions. *Journal of Logic and Computation*, 13(4):469–479, 2003.

[29] P. Hájek and D. Harmancová. Medical fuzzy expert systems and reasoning about beliefs. In M. S. Pedro Barahona and J. Wyatt, editors, *Artificial Intelligence in Medicine*, pages 403–404, Berlin, 1995. Springer.

[30] J. Y. Halpern. *Reasoning About Uncertainty*. MIT Press, 2005.

[31] C. L. Hamblin. The modal 'probably'. *Mind*, 68:234–240, 1959.

[32] R. Keefe. *Theories of Vagueness*. Cambridge University Press, 2000.

[33] I. Leuştean and A. Di Nola. Łukasiewicz logic and MV-algebras. In P. Cintula, P. Hájek, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic - Volume 2*, volume 38 of *Studies in Logic, Mathematical Logic and Foundations*, pages 469–583. College Publications, London, 2011.

[34] J. Łukasiewicz and A. Tarski. Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, cl. III*, 23(iii):30–50, 1930.

[35] E. Marchioni. Possibilistic conditioning framed in fuzzy logics. *International Journal of Approximate Reasoning*, 43(2):133–165, 2006.

[36] D. Mundici. *Advanced Łukasiewicz Calculus and MV-Algebras*, volume 35 of *Trends in Logic*. Springer, New York, 2011.

[37] V. Novák, I. Perfilieva, and J. Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer, Dordrecht, 2000.

[38] A. Tarski. *Logic, Semantics, Metamathematics*. Hackett, Indianapolis, IN, 1983.

[39] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

Ing. Petr Cintula, Ph.D.

Date of birth: August 13, 1978   Web: http://www.cs.cas.cz/cintula/

Education and qualification

2013, Privatdozent, Computational Logic, Vienna Uni. of Technology
2005, Ph.D., Mathematical Engineering, Czech Technical Uni. in Prague
2001, Ing., Software Engineering, Czech Technical Uni. in Prague

Employment history

2001–present	Institute of Computer Science, AS CR senior scientist (2013–present); scientist (2005–13); Ph.D. student (2001–05)
2013	University of Bern, visiting researcher (May–June)
2011–2012	Vienna University of Technology, visiting researcher (August–April)

Research interests

Mathematical fuzzy logic and formal fuzzy mathematics; Abstract algebraic logic; Substructural and many-valued logics

Basic scientometric data

- 31 journal papers, 3 book-chapters, and 10 proceeding papers
- Citations: 350+ (Web of Science), 1400+ (Google Scholar)
- H-index: 13 (Web of Science), 21 (Google Scholar)
- 80+ talks at conferences (6 of them invited)

Awards

Young researchers' award for outstanding achievements (2007) and Otto Wichterle award (2005) bestowed by the Academy of Sciences of the Czech Republic; Junior Scientist Award bestowed by the Learned Society of the Czech Republic (2006); 1st class award of the rector of the Czech Technical University for an outstanding Ph.D. thesis (2005)

Funding ID

Principal (co-)investigator of four grants of GAČR (*Mathematical Fuzzy Logic in Computer Science, Center of Excellence–Institute for Theoretical Computer Science, A Multivalued Approach to Optima and Equilibria in Economics,* and *Logical Foundations of Semantics*) and two of GAAV (*Dynamic Formal Systems* and *Formal Theories of Mathematical Structures with Vagueness*)

Teaching experience:

- Two courses at Charles University (Mathematical Fuzzy Logic and General Theories of Logical Systems); two at Czech Technical University (Foundations of Fuzzy Logic and Applied Non-Classical Logics); and one at Vienna University of Technology (Fuzzy Logic)
- Seven tutorials presented at international events (e.g. *Gentle Introduction to Mathematical Fuzzy Logic*, ESSLLI, Tübingen, 2014)
- Three master theses supervised

Additional professional activities

- Editor of *Archive for Mathematical Logic*, member of the editorial board of *Journal of Applied Logics, Fuzzy Sets and Systems,* and *Soft Computing*
- Co-editor of four books: *Handbook of Mathematical Fuzzy Logic* (two volumes); *Understanding Vagueness: Logical, Philosophical and Linguistic Perspectives*; and *Witnessed Years: Essays in Honour of Petr Hájek*
- Co-editor of five journal special issues (3x FSS, JLC, IGPL)
- Chair of the Council of the Institute of Computer Science, 2012–present; member since 2007
- Coordinator of the working group of Mathematical Fuzzy Logic (more than 100 members from more than 20 countries)
- Chair of Organizing Committee of *ManyVal 2013*, Prague 2013; *Logic Algebra and Truth Degrees*, Prague 2010; *Logical Models of Reasoning with Vague Information*, Čejkovice 2009
- Chair of Programme Committee of *Non-classical Modal and Predicate Logics*, Guangzhou 2011; *Linz Seminar*, Linz 2010; and member at 8 other conferences

Top 5 publications

- PC, C. Noguera. A Henkin-Style Proof of Completeness for First-Order Algebraizable Logics. To appear in *J. Symbolic Logic,* 2015
- PC, C. Noguera. *A general framework for mathematical fuzzy logic.* In P. Cintula, P. Hájek, C. Noguera (eds.), Handbook of Mathematical Fuzzy Logic, pp. 103–207. College Publications, 2011
- P. Hájek, PC. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic,* 71:863–880, 2006
- PC. Weakly Implicative (Fuzzy) Logics I: Basic Properties. *Archive for Mathematical Logic*, 45:673–704, 2006
- L. Běhounek, PC. Fuzzy Class Theory. *Fuzzy Sets and Systems,* 154:34–55, 2005