České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Ing. Tomáš Kroupa, Ph.D.

Modely teorie her v rámci Łukasiewiczova kalkulu

Game-Theoretic Models in the Frame of Łukasiewicz Calculus

Summary

Łukasiewicz logic is an important many-valued generalization of classical logic. Łukasiewicz calculus based on MV-algebras operates with the continuum of truth values, which can capture a rich variety of phenomena related to vagueness and uncertainty. States on MV-algebras express axiomatically the notion of the average truth value of a formula. It was proved that every state can be represented by an integral with respect to a unique regular Borel probability measure on the spectral space of the MV-algebra. Game theory acted as one of the stimuli for the research in states and measures on MV-algebras. In the lecture we survey main results about game-theoretic models on MV-algebras. In particular, we show that Łukasiewicz calculus represents a uniform frame for the study of most types of coalition games and their solutions. We discuss an open problem, which concerns existence of finitely-supported mixed strategy equilibria in the games with payoffs expressed as formulas in Łukasiewicz logic.

Souhrn

Łukasiewiczova logika je důležitým vícehodnotovým zobecněním klasické logiky. Kontinuum pravdivostních stupňů umožňuje v Łukasiewiczově kalkulu, který je založen na MV-algebrách, modelovat pestrou třídu fenoménů souvisejících s vágností a nejistotou. Stav na MV-algebře je axiomatickým vyjádřením průměrné pravdivostní hodnoty formule. Platí, že každý stav lze reprezentovat integrálem vzhledem k jednoznačně určené regulární Borelovské pravděpodobnostní míře na spektrálním prostoru MV-algebry. Jedním ze stimulů k výzkumu stavů a měr na MV-algebrách byla i teorie her. V přednášce shrneme hlavní výsledky v oblasti modelů teorie her na MV-algebrách. Ukážeme, že Łukasiewiczův kalkul představuje jednotný rámec pro studium většiny koaličních her a jejich řešení. Zmíníme i jeden otevřený problém, který se týká existence smíšených rovnovážných strategií s konečným nosičem pro hry, v nichž je užitek modelován formulemi Łukasiewiczovy logiky.

Klíčová slova

Łukasiewiczova logika, MV-algebra, stav, integrální reprezentace, kooperativní hry, Nashova rovnováha

Keywords

Łukasiewicz logic, MV-algebra, state, integral representation, cooperative games, Nash equilibrium

Contents

1 Łukasiewicz calculus

Łukasiewicz logic is among the non-classical logical systems whose rapid development was made possible by current advances in mathematical fuzzy logic (see Hájek's book [11]) and theory of MV-algebras [8], which provide an algebraic semantics of Łukasiewicz logic. Unlike boolean logic based on values 0 and 1, Łukasiewicz logic evaluates formulas in the whole real unit interval $[0, 1]$. Enlarging the scope of truth valuations opens a surprisingly vast field for mathematical discoveries [18], for example:

- Γ functor provides a categorical equivalence between MV-algebras and unital lattice ordered abelian groups having a distinguished order unit.
- Every free finitely generated MV-algebra is isomorphic to the MV-algebra of the so-called McNaughton functions, which are many-valued counterparts of boolean functions.
- Existence of normal forms of Łukasiewicz formulas, which is based on the algebraic-geometric machinery of regular triangulations of the unit cube.
- There exists a one-to-one correspondence between averaging processes (states) of an MV-algebra and regular Borel probability measures on the spectral space of the MV-algebra.

Łukasiewicz calculus was employed to capture several interesting game-theoretical problems, such as Ulam game [16], de Finetti coherence criterion [18], and the Gile's game [9]. The main aim of this lecture is to promote the use of Łukasiewicz calculus in game theory farther: we will show that its tools can be successfully applied to model selected cooperative games (Section 4) and some strategic games (Section 5).

2 MV-algebras

MV-algebras were introduced by Chang [7] as the equivalent algebraic semantics for the infinite-valued Łukasiewicz calculus. In this sense MV-algebras play the same role in Łukasiewicz logic as boolean algebras play in classical two-valued logic.

Definition 2.1. *An* MV-algebra *is an algebra* $\langle M, \oplus, \neg, 0\rangle$ *with a binary operation* $\oplus$*, a unary operation* $\neg$ *and a constant* 0 *such that* *the reduct* $\langle M, \oplus, 0\rangle$ *is an abelian monoid and the following equations hold true for every* $a, b \in M$*:*

$$\neg\neg a = a,$$
$$a \oplus \neg 0 = \neg 0,$$
$$\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a.$$

We put

$$1 = \neg 0, \qquad a \odot b = \neg(\neg a \oplus \neg b).$$

For any two elements $a, b \in M$ we write

$$a \preceq b \quad \text{if} \quad \neg a \oplus b = 1.$$

The relation $\preceq$ is a partial order over M. Further, the operations $\vee, \wedge$ defined by

$$a \vee b = \neg(\neg a \oplus b) \oplus b, \qquad a \wedge b = \neg(\neg a \vee \neg b),$$

respectively, make the algebraic structure $\langle M, \wedge, \vee, 0, 1\rangle$ into a distributive lattice with bottom element 0 and top element 1. We say that M is *semisimple* whenever for every $n \in \mathbb{N}$ and every $a, b \in M$ such that $\bigoplus_{i=1}^{n} a \preceq b$, we have $a \odot b = a$.

Example 2.1 (Standard MV-algebra)**.** *The basic example of an MV-algebra is the* standard MV-algebra, *which is the real unit interval* $[0, 1]$ *equipped with operations*

$$a \oplus b = \min(1, a + b), \qquad a \odot b = \max(0, a + b - 1), \qquad \neg a = 1 - a.$$

The partial order $\preceq$ *of the standard MV-algebra coincides with the usual order of reals* $\leq$ *from the unit interval* $[0, 1]$*.*

Example 2.2. *Every boolean algebra* $\langle A, \vee_A, \wedge_A, \neg_A, 0_A, 1_A\rangle$ *becomes an MV-algebra upon setting* $\oplus = \vee = \vee_A$*,* $\odot = \wedge = \wedge_A$*,* $\neg = \neg_A$*,* $0 = 0_A$*, and* $1 = 1_A$*.*

MV-algebras generalize boolean algebras in the following sense [8, Corollary 1.5.5]: an MV-algebra M is a boolean algebra if and only if the operation $\oplus$ satisfies $a \oplus a = a$, for every $a \in M$. Hence MV-algebras are particular non-idempotent generalizations of boolean algebras. Among other non-boolean algebraic models are orthomodular lattices used in quantum logics [22]. However, by contrast with quantum structures, MV-algebras are always distributive lattices.

Let X be a nonempty set. The set $[0,1]^X$ of all functions $X \to [0,1]$ becomes an MV-algebra if the operations $\oplus, \neg$, and the element 0 are defined pointwise. The corresponding lattice operations $\vee, \wedge$ are then the pointwise maximum and the pointwise minimum of two functions $X \to [0,1]$, respectively.

Definition 2.2. *A* clan *over a nonempty set X is a collection M_X of functions $X \to [0,1]$ such that the zero function 0 is in M_X and the following conditions are satisfied:*

(i) if $a \in M_X$, then $\neg a \in M_X$;

(ii) if $a, b \in M_X$, then $a \oplus b \in M_X$.

In particular, the collection M_X of all continuous $[0,1]$-valued functions over a compact Hausdorff space X is a clan. Since the space X is completely regular, this clan is *separating*: for every pair of different points $x, y \in X$ we can find $a \in M_X$ such that $a(x) \neq a(y)$.

It turns out that clans of $[0,1]$-valued continuous functions over some compact space are, in some sense, the most general examples of MV-algebras. This fact can be viewed as a generalization of Stone's representation theorem for boolean algebras, which says that every boolean algebra is isomorphic to an algebra of sets.

Theorem 2.1. *Let M be an MV-algebra. Then the following are equivalent:*

(i) M is semisimple.

(ii) There exists a compact Hausdorff space X and a separating clan M_X of continuous functions over X such that $M \simeq M_X$.

One of the most important examples of separating MV-algebras is the clan of McNaughton functions over $[0,1]^n$ for some $n \in \mathbb{N}$. We say that $f : [0,1]^n \to [0,1]$ is a *McNaughton function* whenever f is

(i) continuous;

(ii) piecewise linear;

(iii) with each linear piece having $\mathbb{Z}$ coefficients.

By $\mathcal{M}_n$ we denote the MV-algebra of all n-variable McNaughton functions.

3 States

Mundici [17] introduced states on MV-algebras with the intent on modeling average truth values of Łukasiewicz formulas.

Definition 3.1. *Let $\langle M, \oplus, \neg, 0\rangle$ be an MV-algebra. A* state *on M is a function $s : M \to [0,1]$ satisfying the conditions $s(1) = 1$ and*

$$s(a \oplus b) = s(a) + s(b) \tag{1}$$

for every $a, b \in M$ such that $a \odot b = 0$.

The condition (1) means additivity with respect to Łukasiewicz sum $\oplus$ since the requirement $a \odot b = 0$ generalizes disjointness of a pair of elements in a boolean algebra. Hence states are faithful analogues of finitely-additive probability measures. Indeed, every finitely additive probability on a boolean algebra is a state as a special case of the above definition. In particular, every Borel probability measure is a state as well.

Proposition 3.1 ([17])**.** *For every state s on an MV-algebra M we have:*

(i) $s(0) = 0$;

(ii) $s(a \oplus b) + s(a \odot b) = s(a) + s(b)$ *for every* $a, b \in M$;

(iii) $s(a \vee b) + s(a \wedge b) = s(a) + s(b)$ *for every* $a, b \in M$;

(iv) If $a, b \in M$ are such that $a \preceq b$, then $s(a) \leq s(b)$.

Example 3.1. *A trivial example of a state on any MV-algebra M is a homomorphism of M into the standard MV-algebra $[0,1]$. This applies specially to the algebra of n-variable McNaughton functions $\mathcal{M}_n$. Let $x \in [0,1]^n$. Then the evaluation mapping*

$$s_x(f) = f(x), \quad f \in \mathcal{M}_n$$

is a state.

Example 3.2. *More generally, let μ be a Borel probability measure in the unit cube $[0,1]^n$. Then the mapping*

$$s_\mu(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu, \quad f \in \mathcal{M}_n$$

is a state on $\mathcal{M}_n$.

The purely formal resemblance of states to probabilities suggested by Proposition 3.1 and Examples 3.1-3.2 is not accidental. All states on semisimple MV-algebras, which are identified with separating clans of continuous functions, arise as integrals w.r.t. Borel probability measures: this was proved independently by the author and Panti.

Theorem 3.1 ([12, 20])**.** *Let M_X be a separating clan of continuous $[0,1]$-valued functions over a compact Hausdorff space X. If s is a state on M_X, then there exists a unique regular Borel probability measure μ on X such that*

$$s(a) = \int_X a(x)\,\mathrm{d}\mu(x), \quad a \in M_X. \tag{2}$$

The integral representation of s can be generalized to additive mappings that are not necessarily normalized and positive. The following result, which follows from Jordan decomposition property, appears in [14].

Theorem 3.2. *If $s : M_X \to \mathbb{R}$ is a bounded mapping additive in the sense of* (1)*, then there exists a unique regular Borel measure μ such that* (2) *is satisfied.*

4 Coalition games on MV-algebras

A mathematical model of the cooperative game in a coalitional form is due to von Neumann and Morgenstern, who derived the coalition function from the minimax value of the strategic game. However, most contemporary researches conceive the (transferable utility) coalition game as a primitive concept without consideration for its relation to any strategic game. Specifically, let $X = \{1, \ldots, n\}$ be a finite set of *players*. Each subset A of X is said to be a *coalition*. We define a *coalition game* to be a function

$$v : 2^X \to \mathbb{R} \quad \text{such that} \quad v(\emptyset) = 0. \tag{3}$$

The real number $v(A)$ is interpreted as the amount of utility available to the players in A, disregarding actions of all the remaining players in coalition $X \setminus A$. A *payoff distribution* is a real n-dimensional vector $x = (x_1, \ldots, x_n)$, each of which coordinates x_i represents a payoff imputed to player $i \in X$. It is usually required that every payoff distribution x must be *feasible*, that is, $\sum_{i \in A} x_i \le v(A)$ for each $A \subseteq X$.

Which sets of payoff distributions are likely to arise in coalition games? Let $\mathcal{G}$ be a class of coalition games. A *solution (on $\mathcal{G}$)* is a set-valued function

$$\sigma : \mathcal{G} \to 2^{\mathbb{R}} \tag{4}$$

sending each game $v \in \mathcal{G}$ to a set of payoff vectors $\sigma(v)$ in $\mathbb{R}^n$. For example, let $\mathcal{G}$ be the set of all coalition games with the fixed player set $X = \{1, \dots, n\}$. Then the solution σ on $\mathcal{G}$ defined by

$$\sigma(v) = \left\{ x \in \mathbb{R}^n \;\middle|\; x \text{ feasible and } \sum_{i \in X} x_i = v(X) \right\} \tag{5}$$

is said to be the *core* [21, Chapter 3]. Other criteria of economic rationality lead to different solution concepts σ, such as the *prenucleolus* or the *Shapley value* [21, Chapter 6 and 8].

What is the relevance of cooperative games to MV-algebras and their states? The definition of coalition game (3) can be relaxed in several ways. We will focus on the generalizations that make possible:

(i) The formation of more general coalitions than those modeled by subsets $A \subseteq X$.

(ii) The involvement of a high number (theoretically infinite) of players in the game.

Both (i) and (ii) are the game properties expressible by replacing the boolean algebra of all coalitions with an MV-algebra. In particular, the clans (Definition 2.2) are the coalitional structures enabling us to represent intermediate membership of players in coalitions. Hence the following definition makes sense.

Definition 4.1. *Let X be a set of players and M_X be a clan over X. A* (coalition) game *on M_X is a bounded function*

$$v : M_X \to \mathbb{R} \quad \textit{such that} \quad v(0) = 0.$$

The class of games on MV-algebras incorporates most classes of coalition games studied in the game-theoretic literature during past decades.

- *Aubin's games with fuzzy coalitions.* Aubin considered finite player set $X = \{1, \dots, n\}$ together with the unit cube $M_X = [0,1]^n$. Each element $a = (a_1, \dots, a_n) \in [0,1]^n$ is called a *fuzzy coalition*, where a_i is thought of as a partial degree of membership of player $i \in X$ in the fuzzy coalition a. The cube $[0,1]^n$ can be identified with the clan of all $[0,1]$-valued functions over the player set X.

A *game with fuzzy coalitions* is then a bounded real function on $[0,1]^n$. Such an extension of coalition games from 2^X to the cube $[0,1]^n$ is based on the idea of *convexification,* which is ubiquitous in game theory. In fact, we may view the set of all coalitions 2^X as the set of all extreme points of the set of all fuzzy coalitions in the cube $[0,1]^n$. Hence each fuzzy coalition is in the convex hull of classical coalitions. For example, Owen used this idea to lift a coalition function (3) from 2^X to $[0,1]^n$ — by way of a multilinear interpolation — in order to facilitate the computation of the Shapley value. Games with the finite player set and the possibility of fuzzy coalition formation have been subject to the growing research — see [2] for a first-hand account.

- *Aumann's ideal coalitions.* Relaxing the finiteness assumption, Aumann and Shapley [1] investigated games with the player set $X = [0,1]$. The coalition game becomes a (non-additive) bounded real function on Borel subsets of $[0,1]$. Strange as it may seem, the continuum of players models games with many homogeneous individuals, each of which has negligible impact on the overall outcome of coalitional behavior. The same authors also came up with the concept of *ideal coalitions*, which they identified with Borel measurable functions $X \to [0,1]$. Clearly, the set of all ideal coalitions forms a clan.

- *Butnariu-Klement's games on tribes.* Butnariu and Klement [3] considered tribes not necessarily containing all the ideal coalitions and they generalized the Aumann-Shapley construction of the value operator to a broader class of coalition structures.

What is a payoff distribution in the framework of coalition games on MV-algebras? Let v be a game on a clan M_X over a player set X. For the time being suppose that v is nonnegative and $v(1) = 1$. The *feasibility* requirement means that any candidate for a payoff distribution, which is just a mapping $m : M_X \to \mathbb{R}$ defined on coalitions $a \in M_X$, ranges in the unit interval $[0,1]$ only. Moreover, any payoff distribution m should fulfill the additivity condition (1), that is, the payoff $m(a \oplus b)$ attributed to two disjoint coalitions $a, b \in M_X$ equals the total of the two coalitions' payoffs.

Clearly, every state s on M_X (Definition 3.1) meets both the feasibility and the additivity condition. The restriction to the $[0,1]$-valued payoffs may, however, seem artificial. This leads to the following concept of payoff distribution.

Definition 4.2. *Let M_X be a clan over a player set X. A* payoff distribution *is a bounded function $m : M_X \to \mathbb{R}$ satisfying* (1) *for every $a, b \in M_X$.*

Every payoff distribution m among coalitions in M_X induces a payoff distribution μ among the players in X such that $m(a)$ is the average value of the payoff μ with respect to the participation rates $a(x)$ of players $x \in X$: this is the content of Theorem 3.2, which says that

$$m(a) = \int_X a \, \mathrm{d}\mu.$$

Once knowing the structure of payoff distributions, we can start investigating solution concepts for games on MV-algebras. Given a coalition game v on M_X, the *core* of v is defined to be the set of payoff distributions

$$\mathcal{C}(v) = \{\, m \mid m(a) \leq v(a) \text{ for every } a \in M_X \text{ and } m(1) = v(1) \,\}. \tag{6}$$

In words, the core is the set of all feasible payoff distributions that are simultaneously feasible and *Pareto efficient*.

Most core solution concepts (the classical core (5), the Aubin's core, the Butnariu-Klement's core etc.) are obtained as special cases of (6), which demonstrates the unifying role of MV-algebras in coalitional game theory. In the series of papers we have elaborated on the selected game-theoretic problems from the MV-algebraic perspective:

- Generalized Möbius transform [14];
- Core of games on clans [13];
- Existence of value operator [4];
- Bargaining schemes [5].

5 Strategic games with McNaughton functions

Imagine the following two-person game [15], which is a continuous variant of the well-known *matching pennies* [19]. The action of each player (Alice and Bob) consists in selection of a real number, x and y, from the unit interval $[0, 1]$. The choices are made secretly and independently, the players try to maximize their expected utility. The payoff of Alice is $1000 \cdot |x - y|$ euros, while Bob gains $1000 \cdot (1 - |x - y|)$ euros. What is the maximum price p Alice is willing to pay for the participation in this game?

The situation can be described as a two-player constant sum game, where the payoff functions are

$$f_1(x,y) = |x-y| \quad \text{and} \quad g_1(x,y) = 1-|x-y|.$$

Interestingly, both f_1 and g_1 are 2-variable McNaughton functions since they correspond to the Łukasiewicz formulas

$$(X \odot \neg Y) \oplus (\neg X \odot Y) \quad \text{and} \quad (X \oplus \neg Y) \odot (\neg X \oplus Y),$$

respectively, where the former is known as the *Chang distance* [8] and the latter is its negation. Other example leading to utilities described by McNaughton functions is the version of a game called *love and hate* [6] in which the payoff function of the first player is

$$f_2(x,y) = \min(|x-y|, 1-|x-y|)$$

and the second payoff function is defined as $g_2(x,y) = 1 - f_2(x,y)$, for every $x, y \in [0,1]$.

This motivates the investigation of a general two-person (Alice and Bob) constant sum game such that:

- The strategy space of each player is the interval $[0,1]$.
- The payoff function of Alice is an arbitrary McNaughton function $f \in \mathcal{M}_2$, the payoff function of Bob is given by $g = \neg f$.
- A *pure strategy* is a point in $[0,1]$ and a *mixed strategy* is a probability measure defined on Borel subsets of $[0,1]$. Every pure strategy $x \in [0,1]$ can be identified with the Dirac measure δ_x and conversely.

The resulting game is denoted by $\mathcal{G}_f$. When Alice and Bob play a pair of mixed strategies (μ,ν), the expected payoff of Alice is

$$s_f(\mu,\nu) = \int_{[0,1]^2} f \,\mathrm{d}(\mu \times \nu).$$

We say that a pair of mixed strategies (μ^*,ν^*) is a *Nash equilibrium* of the game $\mathcal{G}_f$ if

$$s_f(\mu,\nu^*) \leq s_f(\mu^*,\nu^*) \leq s_f(\mu^*,\nu),$$

for every pair of mixed strategies (μ,ν). As a consequence of the Glicksberg's theorem [10], the game $\mathcal{G}_f$ has a Nash equilibrium in mixed strategies for any $f \in \mathcal{M}_2$.

Theorem 5.1 (Glicksberg)**.** *Let strategy sets be compact Hausdorff topological spaces and payoff functions be real and continuous. Then there exists a Nash equilibrium* (μ^*, ν^*)*.*

The proof of the above theorem is based on a crude compactness argument. Therefore, the quest for methods and algorithms to recover Nash equilibria is one of the main problem for various classes of continuous games [23]. In particular, sufficient conditions for the existence of finitely-supported mixed equilibria are sought.

Example 5.1. *In game* $\mathcal{G}_{f_1}$ *both Alice and Bob may randomize over finite strategy subsets. Indeed, one Nash equilibrium pair is* (μ^*, μ^*)*, where* $\mu^* = \frac{1}{2}(\delta_0 + \delta_1)$*. Since*

$$s_{f_1}(\mu^*, \mu^*) = \tfrac{1}{2},$$

Alice won't pay no more than $p = \frac{1}{2} \cdot 1000 = 500$ *euros for the game ticket. There is another pair of Nash equilibria* $(\mu^*, \delta_{1/2})$*, which means that the optimal response of Bob to the Alice's random choice of* 0 *and* 1 *is the constant selection of number* $1/2$*.*

It is an open question whether the game $\mathcal{G}_f$, where f is an arbitrary 2-variable McNaughton function, possesses a Nash equilibrium pair whose mixed strategies are finitely-supported Borel probability measures. So far we have achieved only partial results in this direction. They are based on the simplicial decomposition of the unit square that linearizes a McNaughton function.

Proposition 5.1. *Let* $f \in \mathcal{M}_2$*. Then there is a simplicial complex* $\mathcal{S}_f$*, which is supported by the unit square* $[0,1]^2$*, and such that* f *is linear over each simplex of* $\mathcal{S}_f$*.*

Let $V(\mathcal{S}_f)$ be the set of all vertices of the complex $\mathcal{S}_f$. Our goal is to approximate f by a matrix game whose equilibria will also be the equilibria of the original game $\mathcal{G}_f$. The matrix game will be given by the following requirements:

(i) The strategy sets are nonempty finite subsets $M, N \subset [0,1]$.

(ii) The payoff function of Alice is the restriction of f to $M \times N$, the payoff function of Bob is the restriction of $\neg f$ to $M \times N$.

Let $M = \{ x_1, \ldots, x_m \}$ and $N = \{ y_1, \ldots, y_n \}$. An *$(M \times N)$-grid $\mathcal{S}$* is the one-dimensional simplicial complex with the vertex set $M \times N$ and the line segments having the endpoints of the form

$$(x_i, y_j), (x_i, y_{j+1}) \quad \text{or} \quad (x_i, y_j), (x_{i+1}, y_j).$$

The matrix game $\mathcal{G}_f^{\mathcal{S}}$ corresponding to $\mathcal{G}_f$ and an $(M \times N)$-grid $\mathcal{S}$ is called a *grid game.*

Theorem 5.2. *Let $f \in \mathcal{M}_2$ and $\mathcal{S}_f$ be an associated simplicial complex. If there exists an $(M \times N)$-grid $\mathcal{S}$ such that*

- *$V(\mathcal{S}_f) \subseteq M \times N$ and*
- *each line segment of $\mathcal{S}$ belongs to some simplex of $\mathcal{S}_f$,*

then every Nash equilibrium of the matrix game $\mathcal{G}_f^{\mathcal{S}}$ is a Nash equilibrium of the game $\mathcal{G}_f$. Consequently, the game $\mathcal{G}_f$ has a pair of finitely-supported Nash equilibrium strategies.

For instance, both games $\mathcal{G}_{f_1}$ and $\mathcal{G}_{f_2}$ satisfy the hypothesis of the above theorem. However, there are McNaughton functions such that the associated game does not fulfill the sufficient condition of Theorem 5.2. The standing conjecture is that the class of games with McNaughton payoff functions, for which finitely-supported Nash equilibria exist, is much broader. This topic is subject to an ongoing research.

References

[1] R. J. Aumann and L. S. Shapley. *Values of non-atomic games*. Princeton University Press, Princeton, N.J., 1974. A Rand Corporation Research Study.

[2] Y. Azrieli and E. Lehrer. On some families of cooperative fuzzy games. *International Journal of Game Theory*, 36(1):1–15, 2007.

[3] D. Butnariu and E. P. Klement. *Triangular Norm Based Measures and Games with Fuzzy Coalitions*. Kluwer, Dordrecht, 1993.

[4] D. Butnariu and T. Kroupa. Shapley mappings and the cumulative value for n-person games with fuzzy coalitions. *European Journal of Operational Research*, 186(1):288–299, 2008.

[5] D. Butnariu and T. Kroupa. Enlarged cores and bargaining schemes in games with fuzzy coalitions. *Fuzzy Sets and Systems*, 5(160):635–643, 2009.

[6] V. Capraro and M. Scarsini. Existence of equilibria in countable games: An algebraic approach. *Games and Economic Behavior*, 79(0):163–180, 2013.

[7] C. C. Chang. Algebraic analysis of many valued logics. *Transactions of the AMS*, 88:467–490, 1958.

[8] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning*, volume 7 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2000.

[9] C. G. Fermüller and G. Metcalfe. Giles's game and the proof theory of Łukasiewicz logic. *Studia Logica*, 92(1):27–61, 2009.

[10] I. L. Glicksberg. A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. *Proc. Amer. Math. Soc.*, 3:170–174, 1952.

[11] P. Hájek. *Metamathematics of fuzzy logic*, volume 4 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 1998.

[12] T. Kroupa. Every state on semisimple MV-algebra is integral. *Fuzzy Sets and Systems*, 157(20):2771–2782, 2006.

[13] T. Kroupa. Core of coalition games on MV-algebras. *Journal of Logic and Computation*, 21(3):479–492, 2011.

[14] T. Kroupa. Generalized Möbius transform of games on MV-algebras and its application to Cimmino-type algorithm for the core. In *Optimization Theory and Related Topics*, volume 568 of *Contemporary Mathematics*, pages 139–158. Amer. Math. Soc., Providence, RI, 2012.

[15] T. Kroupa and O. Majer. Nash equilibria in a class of games with McNaughton functions. In preparation, 2013.

[16] D. Mundici. Ulam games, Łukasiewicz logic, and AF C^*-algebras. *Fundamenta Informaticae*, 18(2-4):151–161, 1993.

[17] D. Mundici. Averaging the truth-value in Łukasiewicz logic. *Studia Logica*, 55(1):113–127, 1995.

[18] D. Mundici. *Advanced Łukasiewicz calculus and MV-algebras*, volume 35 of *Trends in Logic*. Springer, 2011.

[19] G. Owen. *Game theory*. Academic Press Inc., San Diego, CA, third edition, 1995.

[20] G. Panti. Invariant measures in free MV-algebras. *Communications in Algebra*, 36(8):2849–2861, 2008.

[21] B. Peleg and P. Sudhölter. *Introduction to the theory of cooperative games*, volume 34 of *Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research*. Springer, Berlin, second edition, 2007.

[22] P. Pták and S. Pulmannová. *Orthomodular structures as quantum logics*, volume 44 of *Fundamental Theories of Physics*. Kluwer Academic Publishers Group, Dordrecht, 1991.

[23] N. Stein, A. Ozdaglar, and P. Parrilo. Separable and low-rank continuous games. *International Journal of Game Theory*, 37(4):475–504, 2008.

Ing. Tomáš Kroupa, Ph.D.

Date of birth: April 10, 1978 E-mail: kroupa@utia.cas.cz

Web: http://staff.utia.cas.cz/kroupa/

Research interests

- Game theory
- Many-valued logics
- Reasoning under uncertainty

Employment history

- Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic since 2002
- Faculty of Electrical Engineering, Czech Technical University in Prague since 2005

Education

- Ph.D. (Mathematical Engineering), Faculty of Electrical Engineering, Czech Technical University in Prague 2005
- Ing. (Information and Knowledge Engineering), Faculty of Informatics and Statistics, University of Economics, Prague 2001

Basic scientometric data

- 14 journal papers, 2 book-chapters and 21 conference papers
- more than 75 citations, h-index=5 (Web of Science)

Awards

- Prize of the Academy of Sciences for young researchers 2012
- Otto Wichterle Award 2009
- Josef Hlávka's Prize 2005

Stays abroad

- Universidade Federal da Bahia, Brazil (1 month) 2013
- University of Haifa, Israel (6 months) 2005
- Johannes Kepler Universität Linz, Austria (4 months) 2003

Grants (principal investigator)

- Many-valued Approach to Optima and Equilibria in Economics, Czech Science Foundation 2012–2014
- Multidimensional Models of Uncertainty, Czech Science Foundation 2009–2011

Editorial work

- area editor (Game Theory) of Fuzzy Sets and Systems
- member of the editorial board of
  - Journal of Multiple-Valued Logic and Soft Computing
  - Soft Computing
  - Kybernetika

Invited lectures

- 34th Linz Seminar (Non-Classical Measures and Integrals), Linz, Austria, 2013
- Algebraic Semantics for Uncertainty and Vagueness, Salerno, Italy, 2011
- ManyVal'10 (Beyond algebraic semantics: bridging intended and formal interpretations of many-valued logics), Varese, Italy, 2010
- Optimization Theory and Related Topics, Haifa, Israel, 2010

Teaching experience

- Information Theory and Coding, Game Theory, Probability and Statistics, Linear Algebra, Mathematical Logic, Basic Calculus