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Casimirovy invarianty Lieových algeber a jejich aplikace

Casimir Invariants of Lie Algebras with Applications

Summary

Casimir invariants and generalized Casimir invariants play an important role in the theory of Lie algebras, in particular in the theory of their representations. They are also crucial for many applications of Lie algebras in modern physics.

We introduce the notion of Casimir invariant as an element of the center of the universal enveloping algebra of the given Lie algebra and indicate its relevance for the theory of representations of Lie algebras, in particular for identification of irreducible representations.

Next, we discuss some of its applications in quantum physics: in the theory of angular momentum; irreducible representations of the Poincaré algebra, i.e. mathematical description of particles in quantum field theory; and Lie–algebraic computation of the hydrogen spectrum.

It turns out that Casimir invariants also can be naturally identified with polynomial invariants of the coadjoint representation of the given Lie algebra. We call the nonpolynomial invariants of the coadjoint representation generalized Casimir invariants. We present two basic computational approaches to their determination, i.e. the infinitesimal one and the method of moving frames, and show an example of an explicit calculation using both methods. We indicate where these objects appear in our own research. Finally, we review several applications of generalized Casimir invariants, e.g. in the problem of identification of Lie algebras and in symplectic mechanics.

Souhrn

Casimirovy invarianty a zobecněné Casimirovy invarianty hrají důležitou úlohu v teorii Lieových algeber, zejména v teorii jejich reprezentací. Jsou též klíčové pro mnoho aplikací Lieových algeber v moderní fyzice.

Zavádíme pojem Casimirova invariantu jako prvku centra univerzální obalové algebry dané Lieovy algebry a nastiňujeme jeho význam pro teorii reprezentací Lieových algeber, zvláště pro identifikaci ireducibilních reprezentací.

Dále diskutujeme některé jeho aplikace v kvantové fyzice: v teorii momentu hybnosti; v popisu ireducibilních reprezentací Poincaréovy algebry, tj. v matematickém popisu částic ve kvantové teorii pole; a výpočet spektra atomu vodíku s využitím Lieových algeber.

Ukazuje se, že Casimirovy invarianty mohou být přirozeným způsobem ztotožněny s polynomiálními invarianty koadjungované reprezentace dané Lieovy algebry. Nepolynomiální invarianty koadjungované reprezentace nazýváme zobecněné Casimirovy invarianty. Představujeme dva základní přístupy k jejich určení, tj. infinitesimální přístup a metodu pohyblivých reperů, a ukazujeme příklad explicitního výpočtu s využitím obou metod. Dále nastiňujeme, kde tyto objekty vystupují v našem vlastním výzkumu. Na závěr zmiňujeme několik aplikací zobecněných Casimirových invariantů, např. v úloze identifikace Lieových algeber nebo v symplektické mechanice.

Klíčová slova: Lieovy algebry, reprezentace, univerzální obalové algebry, Casimirovy operátory, orbity koadjungované reprezentace

Keywords: Lie algebras, representations, universal enveloping algebras, Casimir operators, coadjoint orbits

Contents

1 Introduction

Casimir invariants and their generalizations play an important role in the theory of Lie algebras and in particular of their representations. As we shall see they are also relevant for their applications in physics.

In order to introduce them we shall start with a review of several notions in the theory of Lie algebras. Next, we define Casimir invariants and discuss some of their properties and applications. Finally, we describe their generalization, i.e. the invariants of coadjoint representation, together with some of its applications.

A *Lie algebra* $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ equipped with a bracket, i.e. an antisymmetric bilinear map $[\ ,\ ]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, such that

$$0 \quad = \quad \big[x,[y,z]\big]+\big[y,[z,x]\big]+\big[z,[x,y]\big] \quad \text{(Jacobi identity)} \tag{1}$$

holds for all elements $x, y, z \in \mathfrak{g}$. For the sake of simplicity we shall consider the fields $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ and finite–dimensional Lie algebras only.

A *representation* ρ of a given Lie algebra $\mathfrak{g}$ on a vector space V is a linear map of $\mathfrak{g}$ into the space $\mathfrak{gl}(V)$ of linear operators acting on V

$$\rho\colon \mathfrak{g}\to\mathfrak{gl}(V)\colon x\to\rho(x)$$

such that for any pair x, y of elements of $\mathfrak{g}$

$$\rho([x,y]) = \rho(x)\circ\rho(y) - \rho(y)\circ\rho(x) \tag{2}$$

holds. A subspace W of V is called *invariant* if

$$\rho(\mathfrak{g})W = \operatorname{span}\{\rho(x)w \,|\, x\in\mathfrak{g}, w\in W\} \subseteq W.$$

A representation ρ of $\mathfrak{g}$ on V is

- *reducible* if a proper nonvanishing invariant subspace W of V exists,
- *irreducible* if no nontrivial invariant subspace of V exists,
- *fully reducible* when every invariant subspace W of V has an invariant complement $\tilde{W}$, i.e.

$$V = W\oplus\tilde{W}, \qquad \rho(\mathfrak{g})\tilde{W}\subseteq\tilde{W}. \tag{3}$$

An important criterion for irreducibility of the given representation is

Theorem 1 (Schur Lemma)**.** *Let $\mathfrak{g}$ be a complex Lie algebra and ρ its representation on a finite–dimensional vector space V.*

1. *Let ρ be irreducible. Then any linear operator A on V which commutes with all $\rho(x)$,*

$$[A, \rho(x)] = 0, \qquad \forall x \in \mathfrak{g},$$

has the form $A = \lambda \mathbf{1}$ for some complex number λ.

2. *Let ρ be fully reducible and such that every linear operator A on V which commutes with all $\rho(x)$ has the form $A = \lambda \mathbf{1}$ for some complex number λ. Then ρ is irreducible.*

The *adjoint representation* of the Lie algebra $\mathfrak{g}$ is a linear map from $\mathfrak{g}$ into the space $\mathfrak{gl}(\mathfrak{g})$ of linear operators acting on $\mathfrak{g}$

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})\colon x \to \mathrm{ad}(x)$$

defined for any pair x, y of elements of $\mathfrak{g}$ as

$$\mathrm{ad}(x)\, y = [x, y]. \tag{4}$$

Now we introduce the notion of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ which is defined as a factoralgebra of the tensor algebra of the given Lie algebra $\mathfrak{g}$. Casimir invariants will be defined as certain distinguished elements in $\mathfrak{U}(\mathfrak{g})$.

The *tensor algebra* of the vector space V over the field $\mathbb{F}$ is the vector space

$$\mathcal{T}(V) = \oplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{F} \oplus V \oplus V \otimes V \oplus \ldots \oplus V^{\otimes k} \oplus \ldots$$

equipped with the associative multiplication generated by the multiplication of decomposable elements

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_k) \cdot (w_1 \otimes \ldots \otimes w_l) = v_1 \otimes v_2 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_l.$$

When the vector space V is in addition a Lie algebra $V = \mathfrak{g}$, one may consider a two–sided ideal $\mathcal{J}$ in the associative algebra $\mathcal{T}(\mathfrak{g})$ generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$, i.e.

$$\mathcal{J} = \mathrm{span}\, \{A \otimes (x \otimes y - y \otimes x - [x, y]) \otimes B \,|\, x, y \in \mathfrak{g},\ A, B \in \mathcal{T}(\mathfrak{g})\}.$$

The factoralgebra

$$\mathfrak{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{J} \tag{5}$$

is called the *universal enveloping algebra* of the Lie algebra $\mathfrak{g}$. Universal enveloping algebras are by construction associative algebras, i.e. the notion of universal enveloping algebra allows us to construct an infinite dimensional associative algebra out of any Lie algebra in a canonical way.

The main reason why universal enveloping algebras are useful is the following observation: any representation ρ of a Lie algebra $\mathfrak{g}$ on a (finite–dimensional, for simplicity) vector space V gives rise to a representation $\tilde{\rho}$ of the tensor algebra $\mathcal{T}(\mathfrak{g})$ defined by

$$\tilde{\rho}(x_1 \otimes x_2 \otimes \ldots \otimes x_k) = \rho(x_1) \cdot \rho(x_2) \ldots \rho(x_k).$$

We have $\tilde{\rho}(\mathcal{J}) = 0$. Consequently, $\tilde{\rho}$ defines also a representation $\hat{\rho}$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on the vector space V

$$\hat{\rho}(a) = \tilde{\rho}(A), \qquad a = A \bmod \mathcal{J} \in \mathfrak{U}(\mathfrak{g}), \quad A \in \mathcal{T}(\mathfrak{g}).$$

2 Casimir invariants

The elements of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, i.e. such $c \in \mathfrak{U}(\mathfrak{g})$ that

$$c \cdot a = a \cdot c$$

holds for all $a \in \mathfrak{U}(\mathfrak{g})$, are called *Casimir invariants* of $\mathfrak{g}$. A necessary and sufficient condition for c to be a Casimir invariant is

$$c \cdot x = x \cdot c, \qquad \forall x \in \mathfrak{g} \simeq \mathfrak{g}^{\otimes 1}/\mathcal{J}.$$

We shall consider nontrivial Casimir invariants only, i.e. those different from elements of $\mathbb{F}/\mathcal{J} \simeq \mathbb{F}$.

The first known example of a Casimir invariant was the so–called *quadratic Casimir invariant* constructed by H. Casimir in [3] for any semisimple Lie algebra $\mathfrak{g}$. Let us take any basis $(e_1, \ldots, e_{\dim \mathfrak{g}})$ of $\mathfrak{g}$ and find the basis $(\tilde{e}^1, \ldots, \tilde{e}^{\dim \mathfrak{g}})$ dual with respect to the Killing form on $\mathfrak{g}$ (which is nondegenerate if and only if $\mathfrak{g}$ is semisimple). Then the following element C of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$

$$C = \sum_{k=1}^{\dim \mathfrak{g}} \tilde{e}^k \otimes e_k \tag{6}$$

is a Casimir invariant of $\mathfrak{g}$. The original motivation for H. Casimir came from quantum mechanics: he was looking for differential operators whose eigenfunctions are matrix elements of a given irreducible representation of $\mathfrak{g}$. Shortly after its introduction, the quadratic Casimir invariant was employed by H. Casimir and B. van der Waerden in [4] to prove in a purely algebraic manner Weyl's theorem on full reducibility of representations of semisimple Lie algebras. All higher order Casimir invariants of semisimple Lie algebras were later determined by G. Racah in [8] when he considered the problem of identification of irreducible representations of the given Lie algebra.

The importance of Casimir invariants for the representation theory of complex Lie algebras comes from the Schur lemma, Theorem 1. In any representation ρ we have

$$[\hat{\rho}(c), \rho(x)] = 0, \qquad \forall x \in \mathfrak{g}.$$

Consequently, if the representation ρ is irreducible, $\hat{\rho}(c)$ must be a multiple of the identity operator, $\lambda\mathbf{1}$. The number λ depends on the choice of the representation ρ and the Casimir invariant c. If two irreducible representations ρ_1 on V_1 and ρ_2 on V_2 are *equivalent*, i.e. if a linear transformation $T : V_1 \to V_2$ exists such that

$$\rho_2(x) = T \circ \rho_1(x) \circ T^{-1}, \qquad \forall x \in \mathfrak{g},$$

then necessarily we have $\lambda_1 = \lambda_2$ for the given Casimir invariant c. That means that the eigenvalues of $\hat{\rho}(c)$ can be used to distinguish inequivalent irreducible representations. The operator $\hat{\rho}(c)$ is called *Casimir operator* in the given representation ρ. Very often the terms Casimir invariant and Casimir operator are used interchangeably, their meaning, i.e. an element of the enveloping algebra vs. an operator on the representation space, being clear from the context.

If ρ is fully reducible but not irreducible then we may use the knowledge of Casimir operators of $\mathfrak{g}$ in the decomposition of ρ into irreducible components. In particular, we construct common eigenspaces of all known Casimir operators and we know that each of them is an invariant subspace (not necessarily irreducible in the case of nonsemisimple Lie algebra $\mathfrak{g}$, i.e. in general the values of Casimir invariants may not determine a unique representation).

Casimir invariants are known to exist for certain classes of Lie algebras, e.g. for semisimple ones or Lie algebras with nonvanishing center, including all nilpotent ones. On the other hand some Lie algebras are known to have no nontrivial Casimir invariants.

3 Casimir operators in physics

Casimir invariants are of primordial importance in physics. They often represent such important quantities as angular momentum, elementary particle mass and spin, Hamiltonians of various physical systems etc. Let us now review some of these applications.

Example 3.1. The *angular momentum algebra*

$$\mathfrak{so}(3) = \text{span}\{L_1, L_2, L_3\}$$

has the nonvanishing Lie brackets

$$[L_j, L_k] = \sum_{l=1}^{3} \epsilon_{jkl} L_l. \tag{7}$$

The quadratic Casimir invariant (6) is

$$C = -\frac{1}{2}\sum_{l=1}^{3} L_l^2, \tag{8}$$

i.e. it coincides up to a numerical factor 1/2 with the square of angular momentum, familiar from the construction of irreducible representations of the angular momentum algebra in quantum mechanics.

Example 3.2. The *Poincaré algebra* $\mathfrak{iso}(1,3)$ is spanned by $M^{\mu\nu}, P^\mu$, $\mu, \nu = 0, \ldots, 3$, with the nonvanishing commutation relations

$$\begin{aligned} [M^{\mu\nu}, P^\rho] &= \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu, \\ [M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}, \end{aligned} \tag{9}$$

where η is the Minkowski metric $\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. We denote by $\epsilon_{\mu\nu\rho\sigma}$ the covariant totally antisymmetric tensor.

There are two independent Casimir invariants of $\mathfrak{iso}(1,3)$, which are usually expressed as

$$P^2 = \sum_{\mu=0}^{3} \eta_{\mu\nu} P^\mu P^\nu \quad \text{and} \quad W^2 = \sum_{\mu=0}^{3} \eta_{\mu\nu} W^\mu W^\nu$$

where the quadruplet of quadratic elements of $\mathfrak{U}(\mathfrak{g})$

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$$

is called the Pauli–Lubanski vector. In this case one of the Casimir invariants is of 2nd order in generators whereas the other is of 4th order. These two Casimir invariants are essential in the construction and labelling of irreducible representations of the Poincaré algebra $\mathfrak{iso}(1,3)$ in relativistic quantum field theory. Notice that in this case one considers infinite dimensional anti–selfadjoint representations of $\mathfrak{iso}(1,3)$.

Example 3.3. *Energy spectrum of hydrogen in quantum mechanics* The Hamiltonian of an electron in hydrogen atom is

$$\hat{H} = \frac{1}{2M} \sum_j \hat{P}_j \hat{P}_j - \frac{Q}{r}, \tag{10}$$

where $\hat{P}_j = -\mathrm{i}\hbar \frac{\partial}{\partial x_j}$ are operators of linear momenta in $\mathbb{R}^3$ with the coordinates x_1, x_2, x_3, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, M is the mass of the electron and $Q = \frac{e^2}{4\pi\epsilon_0}$ in SI units.

The Hamiltonian (10) has three obvious integrals of motion, namely the angular momenta

$$\hat{L}_j = \frac{1}{\hbar} \sum_{k,l} \epsilon_{jkl} \hat{X}_k \hat{P}_l,$$

and three less obvious integrals of motion, namely the components of the Laplace–Runge–Lenz vector

$$\hat{K}_i = \frac{1}{2MQ} \sum_k \sum_j \epsilon_{ikj} (\hat{P}_k \hat{L}_j + \hat{L}_j \hat{P}_k) - \frac{1}{\hbar} \frac{x_i}{r}. \tag{11}$$

The operators $\hat{L}_j, \hat{K}_j$ satisfy the following commutation relations

$$[\hat{L}_j, \hat{L}_k] = \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \hat{L}_l, \qquad [\hat{L}_j, \hat{K}_k] = \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \hat{K}_l,$$

$$[\hat{K}_j, \hat{K}_k] = -\frac{2\mathrm{i}}{MQ^2} \sum_{l=1}^{3} \epsilon_{jkl} \hat{L}_l \hat{H}. \tag{12}$$

That means that they form a Lie algebra on any given energy level, i.e. on a subspace $\mathcal{H}_E$ of the Hilbert space $\mathcal{H}$ consisting of all eigenvectors of $\hat{H}$ with the given energy E. When $E < 0$ the Lie algebra is isomorphic to the Lie algebra $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. The two independent Casimir operators of $\mathfrak{so}(4)$ can be expressed as

$$C_1 = \frac{1}{4} \sum_{j=1}^{3} \left(\hat{L}_j + \sqrt{\frac{MQ^2}{2|E|}} \hat{K}_j \right)^2, \quad C_2 = \frac{1}{4} \sum_{j=1}^{3} \left(\hat{L}_j - \sqrt{\frac{MQ^2}{2|E|}} \hat{K}_j \right)^2. \tag{13}$$

In our representation defined by operators $\hat{L}_j, \hat{K}_j$ the difference of these two Casimir operators vanishes identically

$$C_1 - C_2 = \sqrt{\frac{MQ^2}{2|E|}} \sum_{j=1}^{3} \hat{L}_j \hat{K}_j = 0.$$

The quadratic Casimir operator of $\mathfrak{so}(4)$ is the sum $C = C_1 + C_2$. Using the theory of composition of independent angular momenta it can be derived that in any irreducible representation of $\mathfrak{so}(4)$ the operator C must take the form $2p(p+1)\mathbf{1}$ for some nonnegative integer or half–integer constant p. The equation (13) then implies a relation between p and the energy E of the form

$$2p(p+1) = -\frac{MQ^2}{4E}\left(\frac{2E}{MQ^2} + \frac{1}{\hbar^2}\right) \tag{14}$$

which is just a different formulation of the celebrated Rydberg formula

$$E = -\frac{MQ^2}{2\hbar^2}\frac{1}{(2p+1)^2} \tag{15}$$

where the parameter p is usually replaced by an integer $n = 2p+1 > 0$.

To sum up, we have just seen that the spectrum of hydrogen atom can be derived using the theory of Lie algebras, without need for an explicit construction of eigenfunctions. The original algebraic derivation of the hydrogen spectrum, which is essentially equivalent to the one reviewed here, was presented by W. Pauli in [7]. It preceeded both the discovery of Schrödinger equation and considerations of H. Casimir and was based solely on W. Heisenberg's matrix mechanics.

4 Generalized Casimir invariants

As was shown by Abellanas and Martinez Alonso in [1], Casimir invariants are in one–to–one correspondence with polynomial invariants characterizing orbits of the coadjoint representation of $\mathfrak{g}$. The invariants of the coadjoint representation are not necessarily polynomials and we shall call the nonpolynomial ones *generalized Casimir invariants*.

For certain classes of Lie algebras, including semisimple Lie algebras, perfect Lie algebras and nilpotent Lie algebras, all invariants of the coadjoint representation are functions of polynomial ones [1]. On the other hand, for many solvable and Levi decomposable Lie algebras their functionally independent invariants of the coadjoint representation cannot be chosen as polynomials, i.e. they can be genuinely generalized Casimir invariants.

Let us consider a connected Lie group G together with its Lie algebra $\mathfrak{g}$. The coadjoint representation Ad^* of the Lie group G is its representation on the vector space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ obtained via transposition of the operators in the adjoint representation

$$\langle Ad^*(g)\phi, y\rangle = \langle \phi, Ad(g^{-1})y\rangle, \qquad \forall x \in G, y \in \mathfrak{g}, \phi \in \mathfrak{g}^*.$$

The coadjoint representation ad^* of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$ is obtained by differentiation of Ad^*. Explicitly, we have

$$\langle \mathrm{ad}^*(x)\phi, y\rangle = -\langle \phi, \mathrm{ad}(x)y\rangle, \qquad \forall x, y \in \mathfrak{g}, \phi \in \mathfrak{g}^*.$$

The *invariants of the coadjoint representation*, i.e. the *generalized Casimir invariants*, are functions on $\mathfrak{g}^*$ which are constant on orbits of the coadjoint representation of the Lie group G on $\mathfrak{g}^*$. Equivalently, the invariants of the coadjoint representation are solutions of the following system of partial differential equations

$$\widehat{E}_k I(e_1, \ldots, e_n) = 0, \qquad k = 1, \ldots, n. \tag{16}$$

where $\widehat{E}_k$ are first order differential operators acting on functions on $\mathfrak{g}^*$, i.e. vector fields,

$$\widehat{E}_k = \sum_{a,b=1}^{n} e_b {c_{ak}}^b \frac{\partial}{\partial e_a}, \qquad 1 \leq k \leq n \tag{17}$$

and ${c_{jk}}^l$ are *structure constants* in the chosen basis $(e_1, \ldots, e_n)$ of $\mathfrak{g}$,

$$[e_j, e_k] = \sum_{l=1}^{n} {c_{jk}}^l e_l.$$

In equation (17) the quantities e_a shall be interpreted as commuting independent variables – the coordinates in the basis of the space $\mathfrak{g}^*$, dual to the algebra $\mathfrak{g}$. Using the canonical isomorphism $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ we can identify them with the basis vectors of $\mathfrak{g}$.

Two basic methods of calculating Casimir and generalized Casimir invariants exist. The first method is an infinitesimal one and amounts to solving the system of first order linear partial differential equations (16) step by step. In each step, one uses the method of characteristics to solve one of the equations of the system (16) and re-expresses the remaining equations in terms of its solution. An obvious disadvantage of this method is that the equations become more complicated (e.g. nonlinear) in each step [6]. Nevertheless, it often leads to explicit solutions if a suitable basis of the Lie algebra $\mathfrak{g}$ was chosen to start with.

The second method is more global in nature; it uses the action of the Lie group G on $\mathfrak{g}^*$. It is an application of Cartan's method of moving frames [2] and its modern formulation is due to M. Fels and P. Olver [5]. It can be divided into the following steps:

1. Integration of the coadjoint action of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$ as given by the vector fields (17) to a (local) action Ψ of the group G.

2. Choice of a (local) section Σ transversal to each of the orbits of the action Ψ.

3. Construction of invariants. For a given point $p \in \mathfrak{g}^*$ we find group elements transforming p into $\tilde{p} \in \Sigma$ by the action Ψ. The point $\tilde{p}$ is the intersection of the orbit through p and the section Σ, i.e. is the same for any choice of p lying on the same orbit. Therefore, its coordinates interpreted as functions of p are invariant under the coadjoint action of G, i.e. any functionally independent subset of them defines invariants of the coadjoint representation.

The difficulty in application of this method lies in its sensitivity to the choice of the section Σ because only for a well–chosen section the resulting algebraic equations can be solved explicitly.

In order to illustrate these concepts in a more illuminating manner let us now perform an explicit computation of (generalized) Casimir invariants of a given solvable Lie algebra using both methods.

Example 4.1. Let us consider the 4–dimensional Lie algebra $\mathfrak{s}$ with the nonvanishing Lie brackets

$$[e_2, e_3] = e_1, \qquad [e_2, e_4] = e_3, \qquad [e_3, e_4] = -e_2. \tag{18}$$

The vector fields (17) are

$$\begin{aligned} \widehat{E}_1 &= 0, & \widehat{E}_2 &= -e_1\partial_{e_3} - e_3\partial_{e_4}, \\ \widehat{E}_3 &= e_1\partial_{e_2} + e_2\partial_{e_4}, & \widehat{E}_4 &= e_3\partial_{e_2} - e_2\partial_{e_3}. \end{aligned} \tag{19}$$

We take $\widehat{E}_2$ as the first vector field to which we apply the method of characteristics. We have

$$\frac{\mathrm{d}e_3}{e_1} = \frac{\mathrm{d}e_4}{e_3}$$

and the invariants of $\widehat{E}_2$ are e_1, e_2 and $\xi = e_3^2 - 2e_1e_4$. Therefore, any Casimir invariant of the algebra $\mathfrak{s}$ must be of the from $J = J(e_1, e_2, \xi)$. When we apply $\widehat{E}_3$ to such J we get

$$\widehat{E}_3 J = e_1 \left(2e_2 \frac{\partial J}{\partial \xi} - \frac{\partial J}{\partial e_2} \right) \tag{20}$$

and we obtain solutions of $\widehat{E}_3 J = 0$ in the form $\eta = e_2^2 + e_3^2 - 2e_1e_4$ and e_1. Both e_1 and η are also annihilated by $\widehat{E}_4$. Altogether, we have found that the algebra $\mathfrak{s}$ has two functionally independent Casimir invariants

$$I_1 = e_1, \qquad I_2 = e_2^2 + e_3^2 - 2e_1e_4. \tag{21}$$

Let us now perform the same calculation using the method of moving frames. The flows of the vector fields $\widehat{E}_1, \ldots, \widehat{E}_4$ are

$$\begin{aligned}
\Psi^{\alpha_1}_{\widehat{E}_1}(e_1, e_2, e_3, e_4) &= (e_1, e_2, e_3, e_4),\\
\Psi^{\alpha_2}_{\widehat{E}_2}(e_1, e_2, e_3, e_4) &= (e_1, e_2, e_3 - \alpha_2 e_1, \frac{\alpha_2^2}{2} e_1 - \alpha_2 e_3 + e_4),\\
\Psi^{\alpha_3}_{\widehat{E}_3}(e_1, e_2, e_3, e_4) &= (e_1, e_2 + \alpha_3 e_1, e_3, \frac{\alpha_3^2}{2} e_1 + \alpha_3 e_2 + e_4),\\
\Psi^{\alpha_4}_{\widehat{E}_4}(e_1, e_2, e_3, e_4) &= (e_1, e_2 \cos\alpha_4 + e_3 \sin\alpha_4, e_3 \cos\alpha_4 - e_2 \sin\alpha_4, e_4).
\end{aligned}$$

We compose the flows and obtain the action of a generic group element

$$g(\vec{\alpha}) = \exp(\alpha_4 e_4) \exp(\alpha_3 e_3) \exp(\alpha_2 e_2) \exp(\alpha_1 e_1)$$

in the form

$$\Psi(g(\vec{\alpha})) \quad = \quad \Psi^{\alpha_4}_{\hat{E}_4} \circ \Psi^{\alpha_3}_{\hat{E}_3} \circ \Psi^{\alpha_2}_{\hat{E}_2} \circ \Psi^{\alpha_1}_{\hat{E}_1}. \tag{22}$$

We choose a section Σ given by the equations

$$e_2 = 0, \qquad e_3 = 1. \tag{23}$$

The intersection of the section Σ with the orbit $\{\Psi(g(\vec{\alpha}))(p) | \vec{\alpha} \in \mathbb{R}^4\}$ starting from the point $p = (e_1, e_2, e_3, e_4)$ has the following values of α_2, α_3

$$\alpha_2 = \frac{e_3 - \cos\alpha_4}{e_1}, \qquad \alpha_3 = -\frac{e_2 + \sin\alpha_4}{e_1} \tag{24}$$

(generically, i.e. when $e_1 \neq 0$). The coordinates of the intersection

$$\left(e_1, 0, 1, \frac{2e_1e_4 - e_2^2 - e_3^2 + 1}{2e_1} \right) \tag{25}$$

are independent of the remaining two parameters α_1, α_4. That means that we have found using the method of moving frames that two functionally independent functions e_1 and $\frac{2e_1e_4 - e_2^2 - e_3^2 + 1}{2e_1}$ are generalized Casimir invariants. Equivalently, e_1 and $e_2^2 + e_3^2 - 2e_1e_4$, are Casimir invariants of the algebra $\mathfrak{s}$.

In our original research published in [9, 10, 11] we considered three infinite series of nilpotent Lie algebras of particular structure, e.g. model filiform algebras in [10]. For each of these nilpotent Lie algebras we found all solvable Lie algebras whose nilradicals are isomorphic to the given nilpotent algebra. While investigating properties of these nilpotent and solvable algebras, we explicitly constructed the generalized Casimir invariants of all of them. We used the infinitesimal method to compute invariants of Lie algebras constructed in [10] and the method of moving frames in [11]. Finally, in [9], whose nilradical contains the nilradical of [10] as its codimension 2 subalgebra, we were able to deduce the invariants of Lie algebras occuring there from the invariants found in [10]. The resulting invariants of solvable algebras were expressed in terms of polynomial invariants of their nilradicals and were of several types: polynomials; ratios of powers of polynomials (rational or transcendental); or involving nonremovable logarithms.

5 Applications of generalized Casimir invariants

The invariants of the coadjoint representation belong among important characteristics of any given Lie algebra. For instance, their knowledge may help us to distinguish Lie algebras whose nonequivalence may be difficult to establish by other means, as the following example shows.

Example 5.1. Let us consider two real 6–dimensional solvable Lie algebras $\mathfrak{s}_1, \mathfrak{s}_2$ with the nonvanishing Lie brackets

$$\begin{aligned}
\mathfrak{s}_1: \quad & [e_3,e_4]=e_2,[e_3,e_5]=e_1,[e_4,e_5]=e_3,[e_6,e_1]=e_1,\\
& [e_6,e_2]=-e_2,[e_6,e_3]=0,[e_6,e_4]=e_2-e_4,[e_6,e_5]=e_5,\\
\mathfrak{s}_2: \quad & [e_3,e_4]=e_2,[e_3,e_5]=e_1,[e_4,e_5]=e_3,[e_6,e_1]=e_2,\\
& [e_6,e_2]=-e_1,[e_6,e_3]=0,[e_6,e_4]=-e_5,[e_6,e_5]=e_2+e_4.
\end{aligned} \tag{26}$$

These two algebras $\mathfrak{s}_1, \mathfrak{s}_2$ are real forms of a single complex Lie algebra $\mathfrak{s}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{s}_1 \simeq \mathbb{C} \otimes \mathfrak{s}_2$. The question is whether they are equivalent as real algebras or define two distinct real forms of $\mathfrak{s}_{\mathbb{C}}$.

Their independent generalized Casimir invariants can be written as

$$\begin{aligned}
\mathfrak{s}_1: \quad & e_1e_2, \quad e_1^2 \exp\left(\frac{e_3^2-2e_1e_4+2e_2e_5}{e_1e_2}\right),\\
\mathfrak{s}_2: \quad & e_1^2+e_2^2, \quad (e_1^2+e_2^2)\arctan\frac{e_2}{e_1}-e_1e_2-2e_1e_4+2e_2e_5+e_3^2.
\end{aligned}$$

Since no real transformation can convert trigonometric functions into exponentials and vice versa we immediately see that the algebras $\mathfrak{s}_1, \mathfrak{s}_2$ cannot be isomorphic, i.e. must define two different real forms of $\mathfrak{s}_{\mathbb{C}}$.

The invariants of the coadjoint representation play an important role in the Poisson and symplectic geometry. The vector space $\mathfrak{g}^*$ is naturally equipped with a Poisson structure

$$\{f_1, f_2\}(\phi) = \langle \phi, [\mathrm{d}f_1(\phi), \mathrm{d}f_2(\phi)] \rangle, \qquad f_1, f_2 \in C^\infty(\mathfrak{g}^*),\ \phi \in \mathfrak{g}^*, \quad (27)$$

where the use of the isomorphism $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ is again understood. The symplectic leaves of the Poisson structure (27) define many interesting examples of symplectic manifolds. They turn out to be precisely the orbits of the coadjoint action of G on $\mathfrak{g}^*$. The knowledge of the generalized Casimir invariants may allow us to express these symplectic leaves as solutions of a system of algebraic and/of transcendental equations, e.g. as algebraic varieties when all independent invariants are rational.

The invariants of the coadjoint representation together with semiinvariants, i.e. common eigenfunctions of the operators (17) corresponding to nonvanishing eigenvalues [1], are also essential in the construction of Hamiltonian systems completely integrable in Liouville sense on the cotangent bundle (phase space) of Lie groups. For example, Euler equations on spaces dual to Borel subalgebras of simple Lie algebras and their integrals of motion were constructed by V. V. Trofimov in [12].

6 Conclusions

We have introduced the notions of Casimir invariant [3, 4, 8] and generalized Casimir invariant [1] and described their relationship.

We have reviewed some of their applications. We have seen that in the case of quantum systems with symmetries Casimir operators of the underlying symmetry algebra play an essential role as operators of mass, square of angular momentum or other integrals of motion. On the other hand, generalized Casimir invariants are of less relevance in quantum physics because they cannot be easily identified with operators. Nevertheless, they play an important role in symplectic geometry and in construction of integrable systems.

We have indicated where these objects appear in our own research, in [9, 10, 11]. The results obtained there for particular classes of solvable algebras in arbitrary dimension shall be used to formulate and test general conjectures concerning invariants of solvable Lie algebras, whose general structural theory is presently almost nonexistent.

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