České vysoké učení technické v Praze, Fakulta elektrotechnická

Czech Technical University in Prague, Faculty of Electrical Engineering

Ing. Petr Hušek, Ph.D.

Analýza a syntéza Takagi-Sugeno fuzzy systémů

Analysis and Synthesis of Takagi-Sugeno fuzzy systems

Summary

In present days Takagi-Sugeno systems represent the main direction in development of model based design fuzzy control from theoretical point of view. Indeed, by far the largest number of results related to fuzzy control has been published for stability and stabilization of Takagi-Sugeno models. All of them are based on Lyapunov approach, i.e. on searching for an appropriate Lyapunov function candidate. Although quadratic Lyapunov functions with different ways of blending (piecewise or fuzzy) represent the typical choice some results on how to escape from the quadratic framework (e.g to polynomial) are known. The task is usually formulated as a linear matrix inequality (LMI) problem that can be treated by numerically reliable solvers.

Souhrn

V současné době představují z teoretického hlediska Takagi-Sugeno systémy hlavní směr v návrhu fuzzy řízení založeném na modelu. Zdaleka největší množství výsledků mající vztah k fuzzy řízení se týká stability a stabilizace Takagi-Sugeno modelů. Všechny jsou založeny na Ljapunovově přístupu, tedy na hledání vhodné Ljapunovovy funkce. Ačkoli jsou typickou volbou kvadratické funkce s různým způsobem překrytí (po částech nebo fuzzy), jsou známy i výsledky týkající se nekvadratických (např. polynomiálních) funkcí. Úloha je obvykle formulována jako soustava lineárních maticových nerovností, která je řešena numericky spolehlivými algoritmy.

Klíčová slova: fuzzy řízení; stabilita; inteligentní řízení; modelování

Keywords: fuzzy control; stability; intelligent control; modelling

Contents

1. Introduction

In contrast to conventional methods, fuzzy control was initially introduced as a model free design method utilizing linguistic representation of the knowledge given by a human expert [1]. Nevertheless, number of drawbacks, especially absence of systematic procedures for controller synthesis and impossibility to guarantee closed loop stability and robustness, enforce developing of alternative approaches.

Current research is almost exclusively devoted to model based fuzzy control methods. This evolution was facilitated on Takagi-Sugeno (TS) fuzzy models [2] that allow representing a nonlinear system by systematically obtained fuzzy model. A TS fuzzy model is obtained as a nonlinear blending given by membership functions of usually linear submodels using the sector nonlinearity approach [3]. Such approach makes it possible to capture the nonlinearity of the original system by weights holding the convex sum property that allow to apply classical Lyapunov's direct method analysis and controller synthesis. The controller usually matches the structure of the TS model of the plant and is called parallel distributed compensation (PDC) [4].

Typical choice of Lyapunov function candidate is quadratic one [5] since the stability conditions and design procedures can be formulated as Linear Matrix Inequalities (LMI) that can be solved by numerically reliable convex optimization techniques and semi-definite programming [6]. Plenty of improvements have been achieved during last decade focused mainly on relaxations of the original result [5]. Since finding a common quadratic Lyapunov function is only sufficient condition for stability further attempts have been made to reduce the conservatism of the quadratic approach. Piecewise quadratic and fuzzy Lyapunov functions sharing the same structure as the TS model of the plant seem to have the greatest potential [7, 8, 9, 10]. In the latter case the controller loses the PDC structure.

An interesting direction for future research is provided by polynomial fuzzy models where the overall model is given by nonlinear blending of polynomial submodels instead of linear ones [11]. Such models can be obtained via a Taylor-series approximation of a nonlinear function [12]. The derived conditions employing polynomial Lyapunov functions can be checked with semi-definite programming using Sum-of-Squares (SOS) tools [13].

2. Takagi-Sugeno (TS) fuzzy models

The i-th rule (i=1,…,r) of a continuous TS fuzzy model is considered in the following form:

Rule i: If $z_1(t)$ is M_{i1} and … and $z_p(t)$ is M_{ip}, then

$$\begin{aligned}\dot{x}(t) &= A_i x(t) + B_i u(t) \\ y(t) &= C_i x(t)\end{aligned}.$$

The overall system is given as

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} w_i(z(t))\left\{A_i x(t) + B_i u(t)\right\}}{\sum_{i=1}^{r} w_i(z(t))} = \sum_{i=1}^{r} h_i(z(t))\left\{A_i x(t) + B_i u(t)\right\}$$

$$y(t) = \frac{\sum_{i=1}^{r} w_i(z(t)) C_i x(t)}{\sum_{i=1}^{r} w_i(z(t))} = \sum_{i=1}^{r} h_i(z(t)) C_i x(t) \qquad (1)$$

with

$$z(t) = \left[z_1(t), z_2(t), \cdots, z_p(t)\right],$$

$$w_i(z(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t)), \quad h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}, \quad \sum_{i=1}^{r} h_i(z(t)) = 1$$

where $M_{ij}(z_j(t))$ is grade of membership of $z_j(t)$ in the fuzzy set M_{ij}, $\mathbf{x}(t) \in \Re^n$ is the state vector, $\mathbf{u}(t) \in \Re^m$ is the input vector, $\mathbf{y}(t) \in \Re^q$ is the output vector with matrices A_i, B_i and C_i having appropriate dimensions. Vector $\mathbf{z}(t)$ is formed by premise variables that may be functions of the state variables, external variables or time.

Let us note that the TS fuzzy model (1) can exactly represent a nonlinear system

$$\begin{aligned}\dot{x}(t) &= f\left(x(t)\right) + g\left(x(t)\right)u(t) \\ y(t) &= h\left(x(t)\right)\end{aligned}$$

if the state is assumed to lie on a compact set.

2.1 Stability

Stability plays a crucial role in controller design for any system. Although there are many types and definitions of stability only the following two are of interest in case of TS fuzzy systems.

Definition 1 (Stability in the sense of Lyapunov). The origin is a stable equilibrium point of an autonomous system (1) ($u(t)$=0) if for any given value $\varepsilon > 0$ there is a number $\delta(\varepsilon, t_0) > 0$ such that if $\|x(t_0)\| < \delta$ then $\|x(t)\| < \varepsilon \ \forall t > t_0$.

Definition 2 (Asymptotic stability). The origin is an asymptotically stable equilibrium point if

1. it is stable in the sense of Lyapunov,
2. there is a number $\delta_2(t_0) > 0$ such that $\|x(t_0)\| < \delta_2(t_0) \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0$.

Whenever $\delta_2 > 0$ can be made arbitrarily large, then the origin is said to be globally asymptotically stable.

Because of the nature of TS fuzzy systems those properties and controller design are investigated through direct method of Lyapunov.

Theorem 1. Let $V : \Re^n \supset D \to \Re$ be a positive definite function in *D* such that $\dot{V}(x) \le 0$ (negative semidefinite) in *D*. Then, the origin is stable in the sense of Lyapunov. Moreover, if $\dot{V}(x) < 0$ (negative definite) in $D - \{0\}$ the origin is asymptotically stable. In addition, if $D = \Re^n$ and $V(x)$ is radially unbounded $\left(\|x\| \to \infty \Rightarrow V(x) \to \infty\right)$ the origin is globally asymptotically stable. ♣

2.2 Quadratic Lyapunov function

The natural choice of Lyapunov function is a quadratic one, $V(x) = x^T(t)Px(t),\ P > 0$.

2.2.1 Stability analysis

Theorem 2. [5] The equilibrium point of an autonomous fuzzy system (1) is globally asymptotically stable if there exists a common positive definite matrix *P* such that

$$A_i^T P + PA_i < 0,\ i = 1, 2, \ldots, r. \tag{2}$$

♣

The equation (2) is a feasibility LMI problem that can be solved via convex optimization techniques.

2.2.2 Controller synthesis

The most natural fuzzy controller – parallel distributed compensation (PDC) law [4] – consists in designing a local controller for each plant submodel and using the same rules for their blending. The rule base of the PDC is

Rule *i*: If $z_1(t)$ is M_{i1} and … and $z_p(t)$ is M_{ip}, then

$$u(t) = -K_i x(t),\ i = 1,\dots,r .$$

The overall output of the controller is given by

$$u(t) = \frac{\sum_{i=1}^{r} w_i(z(t))K_i x(t)}{\sum_{i=1}^{r} w_i(z(t))} = \sum_{i=1}^{r} h_i(z(t))K_i x(t) . \tag{3}$$

Since PDC and TS fuzzy model of the plant share the rules the weights $h_i(z(t))$ in (1) and (3) are the same.

The closed loop TS fuzzy system given by substituting (3) into (1) yields

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\left\{A_i - B_i K_j\right\}x(t) . \tag{4}$$

Substituting $\Gamma_{ij} = \left(\left\{A_i - B_i K_j\right\} + \left\{A_j - B_j K_i\right\}\right)/2,\ i, j = 1,\dots,r$ the equation (4) can be written as

$$\dot{x}(t) = \sum_{j=1}^{r} h_i(z(t))h_i(z(t))\left\{A_i - B_i K_i\right\} x(t) + 2\sum_{i<j}^{r} h_i(z(t))h_j(z(t))\Gamma_{ij} x(t) . \tag{5}$$

Sufficient stability condition of the closed loop can be formulated as follows:

Theorem 3. [3] The equilibrium of a fuzzy control system (5) is globally asymptotically stable if there exists a common positive definite matrix P such that

$$\Gamma_{ii}^T P + P\Gamma_{ii} < 0,\ i = 1,2,\ldots,r$$
$$\Gamma_{ij}^T P + P\Gamma_{ij} < 0,\ i < j \le r,\ h_i(z(t))h_j(z(t)) \ne 0 \forall t \quad (6)$$

♣

Some relaxations of the previous result can be found in [3].

To find the feedback gains K_i stabilizing the open loop TS fuzzy model the following result can be applied.

Theorem 4. [3] The equilibrium of the closed-loop fuzzy control systems (4) is asymptotically stabilizable if there exist a common positive definite matrix *X* and a set of matrices $M_i,\ i = 1,\ldots,r$ such that

$$A_i X + XA_i^T - B_i M_i - M_i^T B_i^T < 0,\ \forall i$$
$$A_i X + XA_i^T + A_j X + XA_j^T - B_i M_j - M_j^T B_i^T - B_j M_i - M_i^T B_j^T < 0,\ i < j \quad (7)$$

The feedback gains are determined as $K_i = M_i X^{-1}$. ♣

2.3 Piecewise quadratic Lyapunov functions

Classical LMI approach in the previous section relies on the existence of a common quadratic Lyapunov function. Unfortunately, the standard LMI conditions for quadratic stability are often found to be conservative when applied to fuzzy systems. This conservativeness is produced, at least, by two reasons: (1) For purposes of analysis, fuzzy systems are considered as linear-time varying systems, so their state dependence is disregarded; (2) Lyapunov function is required to be quadratic, though many systems do not admit a global quadratic Lyapunov function or it cannot be found via convex optimization techniques.

An approach based on Lyapunov functions that are piecewise quadratic [7, 8] offers a significant relaxation of the classical LMI approach. Piecewise quadratic Lyapunov functions enrich the set of Lyapunov function candidates that can be used when analyzing a TS fuzzy system. Moreover, they allow to use structural information in the rule base to decrease conservatism of the analysis. In addition, affine TS fuzzy systems can be included under this approach, which improves substantially approximation capabilities of the fuzzy model. Since piecewise Lyapunov function-based approach admits an LMI formulation as well, it is computationally solvable by commercially available software.

2.3.1 Stability analysis

At first let us suppose that in the open loop TS fuzzy model (1) the premise variable vector *z*(.) depends linearly on the state variable vector *x*, i.e. $z = z(x) = \left[z_1(x), \dots, z_p(x) \right] = \left[Z_1 x, \dots, Z_p x \right]$ where $Z_i \in \Re^{1\times n}$, $i = 1, \dots, p$.

The method is based on the partitioning of the state space induced by the membership function supports into two sets of regions. The first set is called the operating regimes and it contains regions where $h_l\left(x\right) = 1$, i.e. where only one (*l*-th) subsystem is active. The remaining regions, where at least two submodels are activated, are called interpolation regimes. Because of linear dependency of premise variables on the state the introduced partitioning results in polyhedral collection $\left\{ X_i \right\}_{i\in I} \subseteq \Re^n$ where *I* is set of indices.

For each cell X_i a set *L*(*i*) will be defined as the set of indices of the system matrices used in the interpolation within that cell. Naturally, for operating regimes, *L*(*i*) contains a single element.

Let us consider the following piecewise quadratic Lyapunov function candidate:

$$V(x) = x^T P_i x,\ x \in X_i,\ i \in I. \tag{8}$$

In order to guarantee continuity across cell boundaries the matrices P_i are parameterized as

$$P_i = F_i^T T F_i,\ i \in I \tag{9}$$

where the matrices F_i fulfill the continuity condition

$$F_i x = F_j x,\ x \in \left\{ X_i \cap X_j \right\},\ i, j \in I. \tag{10}$$

To reduce conservatism of the results the method known as S-procedure [6] is applied. In this case it consists in finding matrices E_i satisfying

$$E_i x \geq 0$$

such that for every matrix W_i with non-negative entries $x^T E_i^T W_i E_i x > 0$ holds $\forall x \in X_i,\ i \in I$.

Theorem 5. [7] If there exist symmetric matrix T and symmetric matrices U_i and W_{ik} with non-negative entries such that $P_i = F_i^T T F_i$, $i \in I$ satisfy

$$\begin{aligned} & P_i - E_i^T U_i E_i > 0 \\ & A_k^T P_i + P_i A_k + E_i^T W_{ik} E_i < 0,\ i \in I,\ k \in L(i) \end{aligned} \tag{11}$$

then the origin is asymptotically stable equilibrium of the autonomous system (1) on the domain of attraction $\bigcup_{i \in I} X_i$. ♣

The matrices F_i and E_i, $i \in I$ can be constructed systematically. Stability of closed loop can be tested analogically to quadratic Lyapunov function case.

2.3.2 Controller synthesis

When using piecewise quadratic Lyapunov function approach to synthesize the PDC controller it is convenient to partition the state space in a slightly different way [8]. Define r partitions $\overline{S}_l$, $l = 1,\dots,r$ as follows:

$$\overline{S}_l = S_l \cup \partial S_l,\ l = 1,\dots,r \tag{12}$$

where

$$\begin{aligned} & S_l = \{x : h_l(x) > h_i(x),\ i = 1,\dots,r,\ i \neq l\} \\ & \partial S_l = \{x : h_l(x) = h_i(x),\ i = 1,\dots,r,\ i \neq l\}. \end{aligned} \tag{13}$$

Then the TS fuzzy system (1) can be written as

$$\begin{aligned} & \dot{x}(t) = \left(A_l + \Delta A_l(h)\right)x(t) + \left(B_l + \Delta B_l(h)\right)u(t) \\ & y(t) = \left(C_l + \Delta C_l(h)\right)x(t) \end{aligned} \tag{14}$$

for $x(t) \in \overline{S}_l$ where

$$\begin{aligned} & \Delta A_l(h) = \sum\nolimits_{i \in M_l} h_i \Delta A_{li},\ \Delta B_l(h) = \sum\nolimits_{i \in M_l} h_i \Delta B_{li},\ \Delta C_l(h) = \sum\nolimits_{i \in M_l} h_i \Delta C_{li}, \\ & A_{li} = A_i - A_l,\ B_{li} = B_i - B_l,\ C_{li} = C_i - C_l \\ & M_l = \{i : h_i \neq 0,\ h_l \geq h_i\}. \end{aligned} \tag{15}$$

Piecewise quadratic Lyapunov function candidate similar to (9) is used,

$$V(x) = x^T P_l x,\ x \in \overline{S}_l,\ l = 1,\dots,r$$

with

$$P_l = F_l^T T F_l,\ F_l x = F_j x,\ x \in \left\{ \overline{S}_l \cap \overline{S}_j \right\},\ l, j \in \{1,\dots,r\}. \tag{16}$$

Employing PDC law (3) the following theorem can be stated.

Theorem 6. [8] The closed loop fuzzy system composed from TS fuzzy model of the plant (14) and PDC law (3) is globally asymptotically stable if there exist constants $\varepsilon_l > 0,\ l = 1,\dots,r,$ a symmetric matrix T and a set of matrices $Q_l,\ l = 1,\dots,r$ such that with

$$P_l = \left(F_l^T F_l\right)^{-1} F_l^T T F_l \left(F_l^T F_l\right)^{-1},\ P_l = R_l^{-1},\ l = 1,\dots,r \tag{17}$$

the following LMIs are satisfied

$$P_l > 0,\ \begin{bmatrix} \Omega_l & P_l & Q_l^T \\ P_l & -\varepsilon_l I & 0 \\ Q_l & 0 & -\varepsilon_l I \end{bmatrix} < 0,\ l = 1,\dots,r \tag{18}$$

where

$$\begin{aligned} &\Omega_l = P_l A_l^T + A_l P_l^T + Q_l^T B_l^T + B_l Q_l + \varepsilon_l \left(E_{lA} E_{lA}^T + E_{lB} E_{lB}^T \right) \\ &\left[\Delta A_l(h)\right]\left[\Delta A_l(h)\right]^T \le E_{lA} E_{lA}^T,\ \left[\Delta B_l(h)\right]\left[\Delta B_l(h)\right]^T \le E_{lB} E_{lB}^T \end{aligned}. \tag{19}$$

The controller gain for each local subsystem is given by

$$K_l = -Q_l P_l^{-1},\ l = 1,\dots,r. \tag{20}$$

♣

Matrices E_{lA}, E_{lB} can be easily calculated.

2.4 Fuzzy Lyapunov functions

Lyapunov function that shares its structure with TS fuzzy model of the plant has been used in [9]. Since the algorithms for controller synthesis are for continuous-time TS fuzzy models still complicated and not effective the results will be presented for discrete-time case even for stability analysis.

2.4.1 Stability analysis

Let us consider discrete-time TS fuzzy model

$$x(t+1) = \sum_{i=1}^{r} h_i(z(t))\left\{A_i x(t) + B_i u(t)\right\} \tag{21}$$

and fuzzy Lyapunov function candidate

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(t)) x^T(t) P_i x(t) \tag{22}$$

with each P_i being a positive definite matrix. Let the matrix Υ_{ij} be defined as

$$\Upsilon_{ij} = \begin{bmatrix} P_i & A_i^T G^T \\ GA_i & G + G^T - P_j \end{bmatrix},\ i, j \in \{1, \dots, r\}\ . \tag{23}$$

Theorem 7. [9] If there exist symmetric matrices $P_i > 0$, Q_{ij} and G such that

$$\begin{gathered} \Upsilon_{ij} > Q_{ij} \\ \Psi_j = \begin{bmatrix} Q_{1j} & 0 & \cdots & 0 \\ 0 & Q_{1j} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & Q_{1j} \end{bmatrix} > 0,\ j = 1, \dots, r \end{gathered} \tag{24}$$

then TS fuzzy system (21) is globally asymptotically stable. ♣

2.4.2 Controller synthesis

Here we employ a non-PDC control law

$$u(t) = -\left(\sum_{i=1}^{r} h_i(z(t)) K_i x(t) \right) \left(\sum_{j=1}^{r} h_j(z(t)) P_j x(t) \right)^{-1} x(t) . \qquad (25)$$

Let us define the matrix Υ_{ij}^{k} as

$$\Upsilon_{ij} = \begin{bmatrix} P_i & \left(A_i P_j - B_i K_j\right)^T \\ A_i P_j - B_i K_j & P_k \end{bmatrix}, \; i, j, k \in \{1, \ldots, r\} . \qquad (26)$$

Theorem 8. [10] If there exist symmetric matrices $P_i > 0,\ Q_{ik} > 0\ \ Q_{ij}^{k},\ j > i$ and matrices K_i such that

$$\begin{aligned} &\Upsilon_{ii}^{k} > Q_i^{k},\ i, k \in \{1, \ldots, r\} \\ &\Upsilon_{ij}^{k} + \Upsilon_{ji}^{k} > Q_{ij}^{k},\ i, j, k \in \{1, \ldots, r\} \\ &\Psi^{k} = \begin{bmatrix} 2Q_1^k & \left(Q_{12}^k\right)^T & \cdots & \left(Q_{1r}^k\right)^T \\ Q_{12}^k & 2Q_2^k & \cdots & \vdots \\ \vdots & \vdots & \ddots & \left(Q_{(r-1)r}^k\right)^T \\ Q_{1r}^k & \cdots & Q_{(r-1)r}^k & 2Q_r^k \end{bmatrix} > 0,\ k = 1, \ldots, r \end{aligned} \qquad (27)$$

then the discrete-time closed loop system composed from TS fuzzy model of the plant (21) and non-PDC controller (25) is globally asymptotically stable. ♣

Even though not presented in this work there are plenty of results related to optimal, robust, adaptive or even more advanced nonlinear control paradigms as well as relaxations of the presented procedures. Nevertheless almost all of them are based on one of the above mentioned approaches.

3. Polynomial fuzzy models

Though LMI-based approaches have enjoyed great success and popularity, there still exists a large number of design problems that either cannot be represented in terms of LMIs and even solved via convex optimization techniques, or the results obtained through LMIs are too conservative.

Recently, a polynomial fuzzy models and control framework has been introduced [11, 12] that can be seen as a generalization of the TS fuzzy models. Such models are more effective in representing nonlinear systems. In order to derive stability and stabilization conditions new class of so called polynomial Lyapunov functions has been utilized. The resulted conditions are represented in terms of sum of squares (SOS) that rely on the SOS decomposition of a multivariate polynomial and can be efficiently computed using semidefinite programming [13].

The rules of a polynomial fuzzy model are considered in the following form:

Rule *i*: If $z_1(t)$ is M_{i1} and … and $z_p(t)$ is M_{ip}, then

$$\begin{aligned} \dot{x}(t) &= A_i\left(x(t)\right)\hat{x}\left(x(t)\right) + B_i\left(x(t)\right)u(t) \\ y(t) &= C_i\left(x(t)\right)\hat{x}\left(x(t)\right),\ i = 1,\ldots,r \end{aligned} \tag{28}$$

where $A_i\left(x(t)\right)$ and $B_i\left(x(t)\right)$ are polynomial matrices in $x(t)$ and $\hat{x}\left(x(t)\right) \in \Re^{N\times 1}$ is column vector of monomials in $x(t)$. A monomial in $x(t)$ is a function of the form $x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)\cdots x_n^{\alpha_n}(t)$ where $\alpha_1,\alpha_2,\ldots,\alpha_n$ are nonnegative integers. Therefore, the right hand sight of each consequent part of the polynomial fuzzy model is a multivariate polynomial in the components of $x(t)$.

Standard defuzzification process leads to the overall representation of (28)

$$\begin{aligned} \dot{x}(t) &= \frac{\sum_{i=1}^{r} w_i(z(t))\left\{A_i\left(x(t)\right)\hat{x}\left(x(t)\right) + B_i\left(x(t)\right)u(t)\right\}}{\sum_{i=1}^{r} w_i(z(t))} \\ &= \sum_{i=1}^{r} h_i(z(t))\left\{A_i\left(x(t)\right)\hat{x}\left(x(t)\right) + B_i\left(x(t)\right)u(t)\right\} \\ y(t) &= \sum_{i=1}^{r} h_i(z(t))\left\{C_i\left(x(t)\right)\hat{x}\left(x(t)\right)\right\} \end{aligned} \tag{29}$$

with

$$w_i(z(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t)),\quad h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))},\quad \sum_{i=1}^{r} h_i(z(t)) = 1$$

where $M_{ij}(z_j(t))$ is grade of membership of $z_j(t)$ in the fuzzy set M_{ij}, $\mathbf{x}(t) \in \Re^n$ is the state vector, $\mathbf{u}(t) \in \Re^m$ is the input vector, $\mathbf{y}(t) \in \Re^q$ is the output vector.

Of course, the TS fuzzy model (1) is a special case of the polynomial model (29).

3.1 Sum of Squares

A multivariate polynomial $f(x(t))$ is SOS if there exist polynomials $f_1(x(t)),\ldots,f_m(x(t))$ such that $f(x(t))=\sum_{i=1}^{m} f_i^2(x(t))$. Naturally, $f(x(t))$ being SOS implies $f(x(t))>0$. Even though vice versa statement is not true, numerical experiments seem to indicate that the gap between SOS and nonnegativity is small.

Proposition 1. Let $f(x(t))$ be a polynomial in $x(t)\in\Re^n$ of degree 2*d*. In addition, let $\hat{x}\left(x(t)\right)$ be a column vector whose entries are all monomials in $x(t)$ with degree no greater than *d*. Then, $f(x(t))$ is an SOS iff there exists a positive semidefinite matrix *P* such that

$$f(x(t))=\hat{x}^T\left(x(t)\right)P\hat{x}\left(x(t)\right). \tag{30}$$

An SOS decomposition for $f(x(t))$ can be computed using semidefinite programming, since it amounts to searching for an element *P* in the intersection of the cone of positive semidefinite matrices and a set defined by some affine constraints that arise from (30).

3.2 Stability analysis

To obtain more relaxed stability condition we use a polynomial Lyapunov function candidate in the form

$$V(x(t))=\hat{x}^T\left(x(t)\right)P\left(x(t)\right)\hat{x}\left(x(t)\right) \tag{31}$$

where $P(x(t))$ is a polynomial matrix in $x(t)$.

To simplify the notation, in the sequel, we drop the notation with respect to time *t*.

Theorem 9. [11] The origin is the stable equilibrium (in the sense of Lyapunov) of the autonomous system (29) (with $u(t)=0$) if there exists a symmetric polynomial matrix $P(x)\in\Re^{N\times N}$, polynomials $\varepsilon_1(x)>0$ for $x\neq 0$ and $\varepsilon_{2i}(x)\geq 0\ \forall x,\quad i=1,\ldots,r$ such that

$$\hat{x}^T(x)\left(P(x)-\varepsilon_1(x)I\right)\hat{x}(x) \text{ is SOS} \tag{32}$$

$$-\hat{x}^T(x)\begin{pmatrix} P(x)T(x)A_i(x)+A_i^T(x)T^T(x)P(x) \\ +\sum_{k=1}^{n}\frac{\partial P}{\partial x_k}(x)A_i^k(x)\hat{x}(x)+\varepsilon_{2i}(x)I \end{pmatrix}\hat{x}(x) \text{ is SOS } \forall i \tag{33}$$

where $A_i^k(x)$ denotes the k-th row of $A_i(x)$ and $T(x)\in\Re^{N\times n}$ is a polynomial matrix whose (i,j)th entry is given by

$$T^{ij}(x)=\frac{\partial\hat{x}_i}{\partial x_j}(x)\,. \tag{34}$$

In addition, if (33) holds with $\varepsilon_{2i}(x)>0$ for $x\neq 0$ then the equilibrium is asymptotically stable. If $P(x)$ is a constant matrix, then the stability holds globally. ♣

Following the procedure for common quadratic Lyapunov function the SOS conditions for PDC controller synthesis of polynomial fuzzy models can be derived.

4. Conclusions

In last two decades Takagi-Sugeno fuzzy models have taken the leading role in fuzzy logic approach to model based control systems. On one hand the ability to exactly represent a large class of nonlinear systems and on the other hand the possibility of using tools for analysis and synthesis known from classical control theory make them attractive tool for dealing with nonlinear control problems. The basic procedures for stability analysis and controller synthesis are based on Lyapunov theory consisting in searching for an appropriate Lyapunov function candidate. Huge amount of results was published on different kinds of Lyapunov functions as common quadratic, piecewise quadratic or fuzzy ones. The common feature for all of them is that the conditions are expressed in the form of LMIs that are solvable by reliable algorithms based on convex optimization techniques. Recently, polynomial fuzzy models were introduced as an enhancement of fuzzy systems with linear submodels in the rules consequents. Even still in the early stages the investigation of those models seems to bring very promising results relying on among control theorists increasingly popular sum of squares techniques.

5. References

[1] Mamdami, E. and S. Assilian: An experiment in linguistic synthesis with a fuzzy logic controller, International Journal of Man-Machine Studies, 7 (1975), 1–13

[2] Takagi, T. and M. Sugeno: Fuzzy identification of systems and its applications to modeling and control, IEEE Transactions on Systems, Man and Cybernetics, 15 (1985), 116–132

[3] Tanaka, K. and H.O. Wang: Fuzzy control systems design and analysis, John Wiley & Sons, New York, 2001

[4] Wang, H.O., K. Tanaka, and M. Griffin: Parallel distributed compensation of nonlinear systems by Takagi-Sugeno fuzzy model, Proc. IEEE Intl. Conf. on Fuzzy Systems ’95, 1995, 531–532

[5] Tanaka, K. and M. Sugeno: Stability analysis and design of fuzzy control systems, Fuzzy Sets and Systems, 45 (1992), 135–156

[6] Boyd, S., L.E. Ghaoui, E. Feron, and V. Belakrishnan: Linear matrix inequalities in system and control theory, vol. 15, SIAM: Studies In Applied Mathematics, Philadelphia, USA, 1994

[7] Johansson, M., A. Rantzer and K. Arzen: Piecewise quadratic stability of fuzzy systems, IEEE Transactions on Fuzzy Systems, 7 (1999), 713–722

[8] Feng, G.: Controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions, IEEE Transactions on Fuzzy Systems, 11 (2003), 605-612

[9] Tanaka, K., T. Hori, and H.O. Wang: A multiple function approach to stabilization of fuzzy control systems, IEEE Transactions on Fuzzy Systems 11 (2003), 582–589

[10] Guerra, T.M. and L. Vermeiren: LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in Takagi-Sugeno’s form, Automatica 40 (2004), 823–829

[11] Tanaka, K., H. Yoshida, H. Ohtake, and H.O. Wang: A sum of squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems, IEEE Transactions on Fuzzy Systems, 17(2009), 911–922

[12] Sala, A.: On the conservativeness of fuzzy and fuzzy- polynomial control of nonlinear systems, Annual Reviews in Control, 33 (2009), 48–58

[13] Prajna, S., A. Papachristodoulou, and F. Wu: Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach, in Proc. 5th Asian Control Conference, 2004

6. Ing. Petr Hušek, Ph.D.

Petr Hušek, born on February 19, 1971 in Chlumec nad Cidlinou, received the M.S. degree in technical cybernetics and Ph.D. degree in control engineering from Faculty of Electrical Engineering, Czech Technical University in Prague, in 1994 and 2002, respectively. The title of the doctoral thesis was “Analysis of Robust Stability of Polynomials with Polynomic Uncertainty”.

Since 1997 he is with Department of Control Engineering at the same faculty where he is since 2004 an Assistant Professor. In 2002 he participated in development of controllers for Takagi-Sugeno fuzzy systems and their application on real systems at University of Maribor under supervision of Prof. Boris Tovornik. In 2004 he collaborated with Associate Professor János Czekkel at University of Miskolc in analysis of robust stability of systems with parametric uncertainty.

Pedagogic experience

At Department of Control Engineering he lectures the courses on Fuzzy control and Modelling and simulation and leads exercises in the courses on Theory of dynamical systems and Fundamentals of systems control. He has been the specialist supervisor of one graduated Ph.D. student, supervisor of one Ph.D. student and supervisor of more than 30 bachelor and diploma theses.

Significant publications

Hušek, P.: On Parametric Hurwitz Stability Margin of Real Polynomials, Control and Cybernetics, vol. 37, no. 3, 2008, pp. 549-564

Hušek, P. and R. Pytelková: Comments on computing extreme values in “Stability Issues on Takagi-Sugeno Fuzzy Model – Parametric Approach”, IEEE Transactions on Fuzzy Systems, vol. 9, no. 1, 2001, pp. 221-222

Bernal, M., P. Hušek and V. Kučera: Nonquadratic stabilization of continuous-time systems in the Takagi-Sugeno form, Kybernetika, vol. 42, no. 6, 2006, pp. 665-672

Roubal, J., P. Hušek and J. Štecha: Linearization: Students Forget the Operating Point, IEEE Transactions on Education, vol. 53, no. 3, 2010, pp. 413-418

Hušek, P.: PI and PD controller synthesis for fuzzy gain and phase margin specifications, appears in International Journal of Innovative Computing, Information and Control, vol. 7, no. 4, 2011