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Diamantové lemma a PBW vlastnost u kvantových algeber

Diamond lemma and PBW property of quantum algebras

Summary

We consider the nonstandard deformation $U'_q(\mathrm{so}_3)$ of the enveloping algebra $U(\mathrm{so}_3)$ and analyze its center. This algebraic structure differs, as expected, when considered parameter q is/is not primitive root of unity. In the first case the center has structure of a polynomial ring in one variable, it is shown here by using Bergman's Diamond lemma. When $q^n = 1$, the size of the structure increases. We encounter three more Casimir elements which are no more algebraically independent. We analyze the structure of central variety and explore explicit polynomial dependence between Casimir elements. This dependence differs for n having form $2m+1$, $2(2m+1)$ and $4m$.

Shrnutí

Uvažujme nestandardní deformaci $U'_q(\mathrm{so}_3)$ obalové algebry $U(\mathrm{so}_3)$ a analyzujme její centrum. Tato algebraická struktura se výrazně liší, pokud je/není deformační parametr q kořenem jednotky. V prvním případě má centrum strukturu okruhu polynomů v jedné proměnné, jak je zde ukázáno pomocí Bergmannova Diamantového lemmatu. Pokud je $q^n = 1$, velikost centra se zvětšuje. Detekujeme tři další Casimirovy elementy, které již nejsou algebraicky nezávislé. Analyzujeme strukturu centrální variety a objevujeme explicitní polynomiální závislost mezi Casimirovými elementy. Tato závislost se liší pro n mající tvar $2m+1$, $2(2m+1)$ a $4m$.

Klíčová slova: kvantová deformace, centrum, centrální varieta, reprezentace, Lieova algebra, Diamantové lemma

Keywords: quantum deformation, center, central variety, representation, Lie algebra, Diamond lemma

Contents

1 Introduction

One of the basic questions when one considers quantum groups or similar deformation algebra structures is to build the adequate representation theory for these associative algebras. Usually we deal with two main cases which are of different difficulty. When deformation parameter, typically denoted by q, is not root of unity, i. e. when $q^n \neq 1$ for all integer n, this theory is similar in many aspects to the classical case, that is to the case of enveloping algebras of classical Lie algebras. On the other hand, when q *is* a root of unity, classification of finite dimensional representations even in the easiest cases is not easy task. It is not surprising that one must undergo, before building representation theory as a whole, various preparatory steps. One of suitable supporting knowledge which helps in doing such a classification is detailed information about the center of the considered algebra. When q is not root of unity, the center of the universal enveloping algebra of a semi simple Lie algebra is a free polynomial algebra, it means it is isomorphic to the ring of polynomials of one or several variables. When we are in the field of quantum groups, there exists one uniform method which can be used to describe this structure quite explicitly. It is called Harish-Chandra homomorphism (it is an analogy to the non-deformed case, see [1]).

For the example, in the simplest case of the algebra $U_q(\mathrm{sl}_2)$, generated by generators E, F, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$

$$[E,F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

the center is generated by

$$C_q = EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2},$$

the Casimir element (for the proof of this example see [2], theorem 45).

When q is root of unity, say $q^n = 1$, the situation is much more difficult. Center is typically much larger and the central elements satisfy nontrivial polynomial relations [3]. In the case of $U_q(\mathrm{sl}_2)$, there are four more additional elements in the center, namely

$$E^p, \quad F^p, \quad K^p, \quad K^{-p},$$

where $p = n$ if n is odd and $p = n/2$ if n is even. These five elements (together with C_q) are no more algebraically independent. One can compute with the help of induction that

$$\prod_{j=0}^{p-1}(C_q-(q-q^{-1})^{-2}(Kq^{j+1}+K^{-1}q^{-j-1}))=E^pF^p,$$

which means

$$C_q^p+\gamma_1C_q^{p-1}+\ldots+\gamma_{p-1}C_q+(-1)^p(q-q^{-1})^{-2p}(K^p-K^{-p})=E^pF^p,$$

where $\gamma_i \in \mathbb{C}$ are certain complex coefficients. This relation describes the central variety of $U_q(\mathrm{sl}_2)$ in the case when q is root of unity.

The quantum groups are not the only sort of quantum deformations. For example, q-deformation $U_q'(\mathrm{so}_3)$ of the universal enveloping algebra $U(\mathrm{so}_3)$, which does not coincide with the Drinfeld-Jimbo quantum algebra $U_q(\mathrm{so}_3)$ is constructed without using the Cartan subalgebra and roots by deforming Serre-type relations directly. We substitute simply $2 \to [2]_q$ where $[a]_q = (q^a-q^{-a})/(q-q^{-1})$ in cubic defining relations of $U(\mathrm{so}_3)$. As a result we obtain complex associative algebra with unity generated by elements I_{21}, I_{32} satisfying the relations

$$I_{21}^2I_{32}-(q+q^{-1})I_{21}I_{32}I_{21}+I_{32}I_{21}^2=-I_{32},$$

$$I_{21}I_{32}^2-(q+q^{-1})I_{32}I_{21}I_{32}+I_{32}^2I_{21}=-I_{21}.$$

It can be shown that this is isomorphic to an algebra generated by three generators I_1, I_2, I_3 and relations [5]

$$q^{\frac{1}{2}}I_1I_2-q^{-\frac{1}{2}}I_2I_1=I_3,$$

$$q^{\frac{1}{2}}I_2I_3-q^{-\frac{1}{2}}I_3I_2=I_1, \tag{1}$$

$$q^{\frac{1}{2}}I_3I_1-q^{-\frac{1}{2}}I_1I_3=I_2.$$

One can quickly explore the following Casimir element, which belongs to the center of this algebra:

$$C=q^2I_1^2+I_2^2+q^2I_3^2-(q^{\frac{5}{2}}-q^{\frac{1}{2}})I_1I_2I_3.$$

Similarly as in the case of ordinary Hopf quantum groups, one can expect that when q is not root of unity, this element should generate the center of the algebra $U_q'(\mathrm{so}_3)$. However, there is no analogy of Harish-Chandra homomorphism to use when trying to prove such a hypothesis.

2 Diamond lemma

In 1978, M. Bergmann recalled rather deep and forgotten result of Newman (see [6], section 3) from the graph theory, often called Diamond lemma and reformulated it to the theory of associative rings. The original Newman formulation was as follows (see [7]). Let G be an oriented graph. Now suppose that

1) The oriented graph g has descending chain condition. That is, all positively oriented paths in G terminate.

2) Whenever two edges, e and e', proceed from one vertex a of G, there exist positively oriented paths p, p' in G leading from the endpoints b, b' of these edges to a common vertex c. (This is often called "confluence" or "diamond" condition.)

Then every connected component C of G has an unique minimal vertex m_C. This means that every maximal positively oriented path beginning at a point of C will terminate at m_C.

Let us now shortly describe the procedure Diamond lemma is based on in theory of associative rings. Having R associative algebra with 1 over complex numbers, its presentation by a family X of generators and family S of relations, where each relation σ is of the form $W_\sigma = f_\sigma$, where W_σ is monomial (product of elements of X) and f_σ is complex linear combination of monomials, the Diamond lemma states that R has a basis consisting of all irreducible monomials (i. e. on which cannot be applied any of relations from S) if the following condition holds: all ambiguities that arise from S are resolvable, that means, all monomials which can be written as product ABC with either $AB=W_\sigma$ and $BC=W_\tau$, $B\neq 1$, or with $ABC=W_\sigma$ and $B=W_\tau$ $(\sigma\neq\tau)$, reduce to a common value.

3 Example of use

M. Bergmann in his article [7] gave various examples of use of Diamond lemma. Let us mention typical of them.

The problem, as mentioned in [8] is as follows. We study ring R defined by generators a, b, c, d and relations

$$a^2=a, \quad b^2=b, \quad c^2=c, \tag{2}$$

$$(a+b+c)^2=a+b+c. \tag{3}$$

The question is does it follow from these relations that $ab=0$? We can easily answer this question constructing basis of R and seeing how element ab is expressed in this basis. Relation (3) can be rewritten to the form

$$cb=-ab-ba-ac-ca-bc. \tag{4}$$

Now we test if (2) and (4), used as reduction relations, yield unique canonical forms for elements of R. There are five ambiguities we must check:

$$a^3, \quad b^3, \quad c^3, \quad cb^2, \quad c^2b. \tag{5}$$

Whereas first three are clearly resolvable, the last two cases yield to two different expressions:

$$c(bb)=cb=-ab-ba-ac-ca-bc$$

and

$$(cb)b=(-ab-ba-ac-ca-bc)b=-ab^2-bab-acb-cab-bcb=$$

$$-ab-bab-a(-ab-ba-ac-ca-bc)-cab-b(-ab-ba-ac-ca-bc)=$$

$$-ab-bab+a^2b+aba+a^2c+aca+abc-cab+bab+b^2a+bac+bca+b^2c=$$

$$-ab-bab+ab+aba+ac+aca+abc-cab+bab+ba+bac+bca+bc=$$

$$aba+ac+aca+abc-cab+ba+bac+bca+bc.$$

Hence we have

$$-ab-ba-ac-ca-bc=aba+ac+aca+abc-cab+ba+bac+bca+bc.$$

This equality can be rewritten to the form

$$cab=aba+2ac+aca+abc+2ba+bac+bca+2bc+ab+ca. \tag{6}$$

From the equality
$$c(cb) = (cc)b$$
we get the same condition. Using (6) as another reduction relation, all ambiguities (5) are resolvable. However, two new ambiguities arise:
$$c^2ab, \quad cab^2.$$
If we test these two, they reduce to common values, for example
$$(cc)ab = cab = aba + 2ac + aca + abc + 2ba + bac + bca + 2bc + ab + ca,$$
$$c(cab) = \ldots = aba + 2ac + aca + abc + 2ba + bac + bca + 2bc + ab + ca.$$
Therefore the irreducible monomials constitute basis of R. These are those words in a, b, c which no letters occurs twice in succession and contain no subwords cab and cb. Note that all relations have on their right hand sides linear combination of words that are lexicographically smaller than a word on the left hand side. Therefore partial ordering exists which is compatible with the reduction relations, descending chain condition is fulfilled. The word ab is irreducible and nonzero.

4 Application: Center of $U_q'(\mathrm{so}_3)$ when q is not root of unity

Let us return to our original problem concerning center of the algebra $U_q'(\mathrm{so}_3)$ when q is not root of unity. We will make extensive use of Diamond lemma in what follows. The key is to construct different basis of the algebra, namely let us have the set
$$\left\{ C^\gamma I_{21}^\alpha (I_{32}I_{21})^k I_{32}^\beta \,\middle|\, \gamma,\alpha,\beta \geqq 0,\ k \in \{0,1\} \right\}.$$
We claim that this set forms linear basis of the algebra $U_q'(\mathrm{so}_3)$. To verify this claim using Diamond lemma, we have to consider the following rewrite rules:
$$I_{32}I_{21}^2 = (q+q^{-1})I_{21}I_{32}I_{21} - I_{21}^2 I_{32} - I_{32},$$
$$I_{32}^2 I_{21} = (q+q^{-1})I_{32}I_{21}I_{32} - I_{21}I_{32}^2 - I_{21},$$

$$I_{32}I_{21}I_{32}I_{21}=C+[3]_q I_{21}I_{32}I_{21}I_{32}-(q+q^{-1})(I_{21}^2+I_{21}^2I_{32}^2+I_{32}^2),$$
$$I_{32}C=CI_{32},$$
$$I_{21}C=CI_{21}.$$

We see that three ambiguities must be checked, namely

$$I_{32}^2I_{21}^2,\quad I_{32}I_{21}I_{32}I_{21}^2,\quad I_{32}^2I_{21}I_{32}I_{21}.$$

One must verify that these expressions do reduce to a common value. Let's test for example the second one:

$$(I_{32}I_{21}I_{32}I_{21})I_{21}=\Big(C+[3]_q I_{21}I_{32}I_{21}I_{32}-(q+q^{-1})(I_{21}^2+I_{21}^2I_{32}^2+I_{32}^2)\Big)I_{21}=$$
$$CI_{21}+[3]_q I_{21}I_{32}I_{21}I_{32}I_{21}-(q+q^{-1})(I_{21}^3+I_{21}^2I_{32}^2I_{21}+I_{32}^2I_{21})=\ldots=$$
$$[5]_q I_{21}^2I_{32}I_{21}I_{32}-[4]_q I_{21}^3I_{32}^2-[2]_q^2 I_{32}I_{21}I_{32}-[4]_q I_{21}I_{32}^2-$$
$$[2]_q[3]_q I_{21}^3+(q+q^{-1})^2CI_{21}+(q+q^{-1})I_{21}.$$

And the other way:

$$I_{32}I_{21}I_{32}I_{21}^2=I_{32}I_{21}(I_{32}I_{21}^2)=I_{32}I_{21}\Big((q+q^{-1})I_{21}I_{32}I_{21}-I_{21}^2I_{32}-I_{32}\Big)=$$
$$(q+q^{-1})I_{32}I_{21}^2I_{32}I_{21}-I_{32}I_{21}^3I_{32}-I_{32}I_{21}I_{32}=\ldots=$$
$$[5]_q I_{21}^2I_{32}I_{21}I_{32}-[4]_q I_{21}^3I_{32}^2-[2]_q^2 I_{32}I_{21}I_{32}-[4]_q I_{21}I_{32}^2-$$
$$[2]_q[3]_q I_{21}^3+(q+q^{-1})^2CI_{21}+(q+q^{-1})I_{21}.$$

We see that the expressions reduce to a common value. The other two ambiguities one can check similarly.

To prove that the center is equal to the set of all polynomials in C it is sufficient to show that any element X from $U_q'(\mathrm{so}_3)$ of the form

$$X=\sum_{\alpha,k,\beta}p_{\alpha,k,\beta}(C)I_{21}^{\alpha}(I_{32}I_{21})^kI_{32}^{\beta},$$

where $p_{\alpha,k,\gamma}$ are any polynomials and indices α,β run from 0 to any value, index k from 0 to 1, and supposing X is from the center of the algebra, which is equivalent to the condition that

$$[X,I_{21}]=0,\qquad [X,I_{32}]=0, \tag{7}$$

is polynomial in C, that is

$$p_{\alpha,k,\beta}(C)=0$$

for all indices α,k,β where $(\alpha,k,\beta)\neq(0,0,0)$. Because of the basis we have chosen this effectively reduces to computation of commutators

$$[I_{32}^{\beta},I_{21}], \qquad [I_{32}I_{21}I_{32}^{\beta},I_{21}].$$

But one does not need explicit form of these complicated commutators, because one can proceed by induction with respect to degree. The total degree of the element $I_{21}^{\alpha}(I_{32}I_{21})^{k}I_{32}^{\beta}$ is $\alpha+2k+\beta$. It is sufficient to show that the highest coefficients are zero and the lower coefficients then follow by induction. To show that the highest coefficient are zero it is sufficient to compute the simplest (highest degree) terms in (7). The highest degree terms we can get if we modify commutation relations such way that we delete all terms on the right hand sides with degree lower than maximal. We have

$$I_{32}^{\beta}I_{21}\sim[1-\beta]_{q}I_{21}I_{32}^{\beta}+[\beta]_{q}I_{32}I_{21}I_{32}^{\beta-1}$$

and

$$I_{21}^{\alpha}I_{32}I_{21}I_{32}^{\beta}I_{21}\sim[\beta+2]_{q}I_{21}^{\alpha+1}I_{32}I_{21}I_{32}^{\beta}-[\beta+1]_{q}I_{21}^{\alpha+2}I_{32}^{\beta+1}.$$

Now we rewrite the element X to the form

$$X=\sum_{d=0}^{m}\sum_{\substack{\alpha,k,\beta\\ \alpha+2k+\beta=d}} p_{\alpha,k,\beta}(C)I_{21}^{\alpha}(I_{32}I_{21})^{k}I_{32}^{\beta},$$

and compute the commutator of maximum degree part only:

$$\left[\sum_{\alpha+2k+\beta=m} p_{\alpha,k,\beta}(C)I_{21}^{\alpha}(I_{32}I_{21})^{k}I_{32}^{\beta},I_{21}\right]\sim$$

$$\sum_{\alpha+2k+\beta=m} p_{\alpha,k,\beta}(C)\Big(\big([(-1)^{k+1}\beta+1+k]_{q}-1\big)I_{21}^{\alpha+1}(I_{32}I_{21})^{k}I_{32}^{\beta}+$$

$$(-1)^{k}[\beta+k]_{q}I_{21}^{\alpha+2k}(I_{32}I_{21})^{1-k}I_{32}^{\beta-1+2k}\Big).$$

From this system of linear equations follows that X has the form

$$X = \sum_{\alpha} p^*_\alpha(C) I_{21}^\alpha,$$

where p^*_α are certain complex polynomials of one variable. Commuting with I_{32} we easily get that all these polynomials with the exception of absolute term are equal identically to zero and the proof is complete.

5 Center of $U'_q(\mathrm{so}_3)$ when q is root of unity

As we said at the beginning, the case when q is root of unity is much more complicated. For the illustration we first take $q^3 = 1$.

Again, commuting general element of the algebra with generators I_1, I_2, we promptly discover the following four Casimir elements from the center of the algebra: one old common to the case when q is not root of unity and three new Casimir elements of the form

$$\begin{aligned} C_{31} &= I_1^3 + I_1, \\ C_{32} &= I_2^3 + I_2, \\ C_{33} &= I_3^3 + I_3. \end{aligned} \tag{8}$$

Our conjecture, similar to the case of Hopf deformations, is that these four Casimir elements are not algebraically independent. Indeed, we can show this applies also for our case in the following way: the basis of $U'_q(\mathrm{so}_3)$ can now be chosen as

$$\left\{ C_{31}^\alpha C_{32}^\beta C_{33}^\gamma I_1^k I_2^l I_3^m \,\middle|\, \alpha,\beta,\gamma \geqq 0,\ k,l,m \in \{0,1,2\} \right\}. \tag{9}$$

The proof is again easy due to Diamond lemma, we must compute all ambiguities such as $I_2^3 I_1$ etc. and show that they reduce to common values.

Let's now compute the powers C, C^2, C^3 and express these powers in the basis (9). For any power this is a finite sum having max. $3^3 = 27$

terms with coefficients being polynomials in three variables C_{31}, C_{32} and C_{33}. Putting together linear combination

$$\alpha+\beta C+\gamma C^2+\delta C^3$$

and investigating if the coefficients α, β, γ, δ as polynomials in three variables C_{31}, C_{32} and C_{33} can be chosen such that this combination becomes equal to zero leads us to the relation

$$C^3-qC^2-C_{31}^2-C_{32}^2-C_{33}^2+3(q+q^{\frac{1}{2}})C_{31}C_{32}C_{33}=0. \tag{10}$$

For the case $q^3=1$ we've proved our hypothesis.

If we want to generalize the results for arbitrary case $q^n = 1$ (primitive root of unity, $n>2$), the proof of (9) becomes less trivial. We present here only final results of computations.

Casimir elements (8) now take general form (cf. [10] and [9], formula (5))

$$C_{n,k}=\sum_{j=0}^{[(n-1)/2]}\binom{n-j}{j}\frac{n}{n-j}\left(\frac{i}{q-q^{-1}}\right)^{2j}I_k^{n-2j},\ k=1,2,3. \tag{11}$$

Now define

$$P_{2m+1}(x,y,z,w)=x^2+y^2+z^2-q^{m+\frac{1}{2}}(q-q^{-1})^{2m+1}xyz-\prod_{k=0}^{2m}(w+q[k]_q[k+1]_q). \tag{12}$$

and

$$P_{2m}(x,y,z,w)=\left(x+y+z-\prod_{k=0}^{m-1}(w+q[2k]_q[2k+1]_q)\right)^2-$$

$$(q-q^{-1})^{2m}xyz \quad \text{for } m \text{ odd},$$

$$P_{2m}(x,y,z,w)=x^2+y^2+z^2-2(xy+xz+yz)-(q-q^{-1})^{2m}xyz- \tag{13}$$

$$2\left(x+y+z+\frac{8}{(q-q^{-1})^{2m}}\right)\prod_{k=0}^{\frac{m}{2}-1}(w+q[2k]_q[2k+1]_q)^2+$$

$$\prod_{k=0}^{\frac{m}{2}-1}(w+q[2k]_q[2k+1]_q)^4 \quad \text{for } m \text{ even.}$$

Then we have

1. The dimension of algebra $U'_q(\mathrm{so}_3)$*, seen as module over a commutative polynomial ring generated by Casimir elements* $C_{n,1}$*,* $C_{n,2}$ *and* $C_{n,3}$*, is* n^3*.*

2. Let $q^n = 1$ *be primitive root of unity. Then the Casimir elements* C *and* C_{nj}*,* $j = 1,2,3$*, satisfy the relation* $P_n(C_{n,1}, C_{n,2}, C_{n,3}, C) = 0$*, where* P_n *is given by (12) resp. (13).*

6 Conclusions

We have shown the power and simplicity of Diammond lemma when constructing various kinds of bases in quantum algebras. This way we have found various informations about the structure of central variety of given algebras. The technique used is available in standard Hopf algebras as well as in nonstandard deformations where standard methods of proofs such as those using variants of Harish Chandra homomorphism are no more available.

7 References

[1] G. Lusztig: Introduction to Quantum Groups, Birkhäuser, Boston, 1993.

[2] A. U. Klimyk, K. Schmüdgen: Quantum groups and their representations, Springer, Berlin, 1997.

[3] R. Tange: J. Algebra **301**, 425–445 (2006).

[4] A. U. Klimyk: Czech. J. Phys. **51**, 331–340 (2001).

[5] M. Havlíček, A. U. Klimyk, S. Pošta: J. Math. Phys. **40**, 2135–2161 (1999).

[6] M. H. A. Newman: Ann. of Math. **43**, 223–243 (1942).

[7] G. M. Bergmann: Adv. Math. **29**, 178–218 (1978).

[8] Problem 5082, Amer. Math. Monthly **70**, 335 (1963).

[9] M. Havlíček, S. Pošta: J. Math. Phys. **42**, 472–500 (2001).

[10] M. Havlíček, A. U. Klimyk, S. Pošta: Czech J. Phys. **50**, 79–84 (2000).

[11] D. P. Zhelobenko: Compact Lie Groups and Their Representations, Amer. Math. Soc., Providence, RI, 1973.
[12] M. Havlíček, A. U. Klimyk, S. Pošta: J. Math. Phys. **42**, 5389–5416 (2001).

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