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Identity algebry oktonionů

Identities of octonion algebras

Summary

In this talk we formulate known results on identities of octonion algebras. We show how the superalgebra method can be used to describe all skew-symmetric identities and central polynomials of octonions.

Souhrn

V této přednášce zformulujeme známé výsledky o identitách algebry oktonionů. Ukážeme jak metoda superalgeber může být použita k popisu všech antisymetrických identit a centrálních prvků algebry oktonionů.

Klíčová slova: algebra oktonionů, superalgebra, volná algebra, identita, centrální polynom.

Keywords: octonion algebra, superalgebra, free algebra, identity, central polynomial.

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1 Introduction

Octonion algebras play an important role in algebra and geometry and have been studied from different points of view by many authors. Nevertheless, identities of octonions have not been described until now.

We recall that an *octonion algebra* over a field F is an 8-dimensional composition algebra over F. In other words, it is a unital non-associative algebra A over F with a (strictly) nondegenerate quadratic form n such that $n(xy) = n(x)n(y)$ for all $x, y \in A$.

One form of the octonions, the Cayley octonions $\mathbb{O}$ over the reals, has a particularly symmetrical base $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$, with multiplication defined by

$$\begin{aligned} 1e_i = e_i = e_i1, \quad e_i^2 = -1, \\ e_ie_{i+1} = e_{i+3} = -e_{i+1}e_i, \\ e_{i+1}e_{i+3} = e_i = -e_{i+3}e_{i+1}, \\ e_{i+3}e_i = e_{i+1} = -e_ie_{i+3}, \end{aligned}$$

where $1 \leq i \leq 7$, and where subscripts are to be interpreted modulo 7.

We also recall that an element $f(x_1, \ldots, x_n) \in F\{X\}$ of the free non-associative algebra is called an *identity* (or a *polynomial identity*) of the algebra A if

$$f(a_1, \ldots, a_n) = 0 \text{ for any } a_1, \ldots, a_n \in A.$$

All identities of A form a T-ideal, i.e. an ideal of $F\{X\}$ which is invariant under endomorphisms of $F\{X\}$. An element $f(x_1, \ldots, x_n) \in F\{X\}$ is called a *central polynomial* (or a *central function*) for A if it is not an identity of A and $f(a_1, \ldots, a_n)$ belongs to the center of A for any $a_1, \ldots, a_n \in A$. A multilinear element $f(x_1, \ldots, x_n) \in F\{X\}$ is called *skew-symmetric* if, for any $\sigma \in Sym(n)$,

$$f(x_1, x_2, \ldots, x_n) = \mathrm{sign}(\sigma) f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

An exact base for the T-ideal of identities of an octonion algebra was found by Isaev [6] only in the case of a finite field F. Il'tyakov [5] proved that, over a field of characteristic zero, the T-ideal of identities of an octonion algebra is finitely generated, without giving a set of generators.

The identities of small degrees of octonion algebras were studied by Racine [11] and Hentzel and Peresi [4]. We classified all multilinear skew-symmetric identities and central polynomials of an octonion algebra over a field F of characteristic zero. Our main result is the following theorem [21].

Theorem. *Every skew-symmetric identity of an octonion algebra over a field of characteristic zero is a consequence of the following skew-symmetric identities:*

$$\begin{aligned} \sum_{Alt} [x_1, x_2](x_3, x_4, x_5) &= 0, \\ \sum_{Alt} (12[x_1, x_2][x_3, x_4][x_5, x_6] - [x_1, x_2, x_3, x_4, x_5, x_6]) &= 0, \\ \sum_{Alt} ([[x_1, x_2, x_3, x_4], [x_5, x_6]] + [x_1, x_2, x_3, x_4, x_5, x_6]) &= 0. \end{aligned}$$

Every skew-symmetric central polynomial of an octonion algebra is a consequence of the skew-symmetric central polynomials:

$$\sum_{Alt}[x_1, x_2][x_3, x_4],$$
$$\sum_{Alt}(12[x_1, x_2][x_3, x_4]x_5 - [x_1, x_2, x_3, x_4, x_5]).$$

Here $\sum_{Alt}$ means the alternating sum over all of the arguments,

$$(x, y, z) = (xy)z - x(yz)$$

is the associator of x, y, z,

$$[x, y] = xy - yx$$

is the commutator of x, y, and $[x_1, \ldots, x_{n-1}, x_n]$ denotes the "long commutator" of the elements $x_1, \ldots, x_n$ which is defined by induction:

$$[x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

Observe that the first identity $\sum_{Alt}[x_1, x_2](x_3, x_4, x_5) = 0$ has minimal possible degree 5 and hence may be considered as an analogue for octonions of the famous Amitsur-Levitsky skew-symmetric identity

$$S_{2n}(x_1, \ldots, x_{2n}) = \sum_{Alt} x_1 \cdots x_{2n} = 0$$

for the matrix algebra M_n. Moreover, $\sum_{Alt}[x_1, x_2](x_3, x_4, x_5)$ is a non-zero element of a minimal degree from the radical of the free alternative algebra of rank $n > 4$ (see [20]).

It is known that the radical $R = Rad(Alt[X])$ of the free alternative algebra coincides with the set of all nilpotent elements of $Alt[X]$ and is equal to the intersection of the associator ideal of $Alt[X]$ and the T-ideal of identities of an octonion algebra [14], [25]. Till now, the smallest degree for known elements from the radical R was 6. For example, such is famous "Kleinfeld's element" $([x_1, x_2]^2, x_3, x_4)$ or the element $[[x_1, x_2] \circ (x_3, x_4, x_5), x_6]$, where

$$x \circ y = xy + yx$$

is the Jordan product of x, y.

For the proof of our results we used the superalgebra approach to the study of skew-symmetric identities developed in [16], [20]. We constructed a base of the free quadratic alternative superalgebra generated by one odd element and described the ideal of super-identities and the subsuperalgebra of central polynomials of this superalgebra. The standard procedure of taking a Grassmann envelope gave a description of skew-symmetric identities and central polynomials in quadratic alternative algebras. Then we checked that there are no additional identities or central polynomials in octonions.

Quadratic alternative algebras over an arbitrary field of characteristic $\neq 2$ were completely classified by Elduque [1]. They include the quaternion and octonion algebras, but also many other non-associative algebras used in physics. The identities of small degrees of arbitrary quadratic algebras were studied by Hentzel and Peresi in [3].

2 Identities of octonion algebras

In this section we formulate known results on identities of octonions.

Isaev [6] proved that over a field F with q elements the T-ideal of identities of an octonion algebra is generated by the polynomials $f_1(x)$, $f_2(x,y)$, $f_3(x,y,z)$, where

$$\begin{gathered}
f_1(x) = (x - x^q)(x - x^{q^2}),\\
f_2(x,y) = (x - x^q) \circ (y - y^q) - ((x - x^q) \circ (y - y^q))^q,\\
f_3(x,y,z) = g_1(x,y,z) g_2(y,z) g_3(x,y,z),\\
g_1(x,y,z) = (x - x^q)((y - y^{q^2})(z - z^{q^2})),\\
g_2(y,z) = (1 - y^{q^2-q})(1 - z^{q^2-q})(1 - (y \circ z)^{q-1}),\\
g_3(x,y,z) = (1 - (x \circ y)^{q^2-q})(1 - (x \circ z)^{q^2-q})(1 - (x \circ yz)^{q^2-q}),
\end{gathered}$$

and the order of the parentheses in f_3, g_2, g_3 is arbitrary.

Racine [11] studied minimal identities of octonions over an arbitrary field F of characteristic not equal to 2, 3, or 5. He proved that (modulo identities of alternativity) there are no polynomial identities of degree less than 5 of an octonion algebra, and that all polynomial identities of degree 5 of an octonion algebra are consequences of the following two ones:

$$\begin{aligned}
[[x,y]^2, x] &= 0,\\
S_3^+(x_1,x_2,x_3)(x^2) - S_3^+(x_1,x_2,x_3)(x) \circ x &= 0.
\end{aligned}$$

Here $S_3^+(x_1,x_2,x_3) = S_3(V_{x_1}, V_{x_2}, V_{x_3})$, where S_3 stands for the standard associative polynomial of degree 3 and $V_x(y) = y \circ x$.

Hentzel and Peresi [4] continued looking for identities and central polynomials of small degrees of octonion algebras with the aid of a computer. In addition to the known central polynomial of degree 4 (see [25])

$$C_4(x_1,x_2,x_3,x_4) = [x_1,x_2] \circ [x_3,x_4],$$

they found a new multilinear central polynomial of degree 5

$$C_5(x_1,x_2,x_3,x_4,x_5) = \sum_{Alt} (24x_1(x_2(x_3(x_4x_5)))+8x_1((x_2,x_3,x_4)x_5) - 11(x_1,x_2,(x_3,x_4,x_5))).$$

They also proved that there are no new central polynomials of degree 6. Moreover, the only new multilinear polynomial identity of degree 6 is $[C_5(x_1,x_2,x_3,x_4,x_5), x_6] = 0$.

It is well known [25] that octonion algebras are quadratic and alternative. We recall that an algebra is called *alternative* if it satisfies the following identities:

$$\begin{aligned}
(x,x,y) &= 0 \qquad \text{(left alternativity)},\\
(x,y,y) &= 0 \qquad \text{(right alternativity)}.
\end{aligned}$$

A unital algebra A is called *quadratic over F* if each element $x \in A$ satisfies the equality

$$x^2 - \tau(x)x + n(x)1 = 0,$$

where $\tau(x)$ is a linear form (*a trace*) and $n(x)$ is a quadratic form (*a norm*) on A, and $n(1) = 1$ (equivalently, $\tau(1) = 2$).

It is easy to see that the minimal identities of Racine are true in any quadratic alternative algebra. The same is true for the centrality of C_4, but not evident at all for C_5. In this connection, it seems interesting that our results on skew-symmetric identities and central polynomials are also true for any quadratic alternative algebra. Notice that for our central polynomials it holds

$$\begin{aligned}\sum_{Alt}[x_1, x_2][x_3, x_4] &= \tfrac{1}{2}\sum_{Alt} C_4(x_1, x_2, x_3, x_4),\\ \sum_{Alt}(12[x_1, x_2][x_3, x_4]x_5 - [x_1, x_2, x_3, x_4, x_5]) &= 2C_5(x_1, x_2, x_3, x_4, x_5).\end{aligned}$$

3 The superalgebra method

Let A be an algebra over a field F of characteristic zero. It seems natural to try to reduce the number of different variables in the identities.

It is well known that reduction of this kind exists for symmetric multilinear polynomials: every such polynomial on n variables may be obtained by a *linearization* or *polarization* of a polynomial of degree n on one variable.

Now, we assume that $f : A^n \to A$ is a skew-symmetric multilinear function. Take the Grassmann algebra G over F generated by Grassmann variables $e_1, e_2, \ldots$; that is the unital associative algebra over F subject to the relations $e_ie_j = -e_je_i,\ i, j = 1, 2, \ldots$. Form the tensor product $G \otimes A$, and extend the function f to it by setting

$$f_G(g_1 \otimes x_1, \ldots, g_n \otimes x_n) = g_1 \cdots g_n \otimes f(x_1, \ldots, x_n).$$

Then f_G becomes a symmetric function on the variables $y_i = e_i \otimes x_i$; moreover,

$$n!e_1 \cdots e_n \otimes f(x_1, \ldots, x_n) = f_G(z, \ldots, z),$$

where $z = e_1 \otimes x_1 + \cdots + e_n \otimes x_n$. It is clear, for example, that $f(x_1, \ldots, x_n) = 0$ if and only if $f_G(z, \ldots, z) = 0$, and the identity $f(x_1, \ldots, x_n) = 0$ is reduced to an identity in one variable over $G \otimes A$. So, in a skew-symmetric case we also can reduce the number of variables, only the new variables lie not in A but in $G \otimes A$. Notice that a similar trick works also for a symmetric identity which may be reduced to an identity in a single variable $z = e_1e_2 \otimes x_1 + \cdots + e_{2n-1}e_{2n} \otimes x_n$ over $G \otimes A$.

The problem is that in general $G \otimes A$ does not belong to the same variety as A; for instance, if $A = F$ then $G \otimes F = G$ is already not commutative. Nevertheless, $G \otimes A$ satisfies certain *graded identities* related with those of A.

Recall that, in general, a *superalgebra* means a $\mathbb{Z}_2$-graded algebra, that is an algebra A which may be written as a direct sum of subspaces $A = A_0 + A_1$ subject to the relations $A_iA_j \subseteq A_{i+j \pmod 2}$. The subspaces A_0 and A_1 are called the *even* and the *odd* parts of the superalgebra A; and so are called the elements from A_0 and from A_1, respectively. Below all the elements are assumed to be homogeneous, that is, either even or odd, and for an element $x \in A_i,\ i \in \{0, 1\}$, the symbol $\bar{x} = i$ denotes its parity.

The Grassmann algebra G has a base over F consisting of 1 together with all the possible products $e_i e_j \cdots e_k$ with $1 \le i < j < \ldots < k$. It can be considered as a superalgebra $G = G_0 + G_1$, where G_0 is spanned by the products of even length, and G_1 is spanned by the products of odd length. Then $G_i G_j \subseteq G_{i+j \ (mod\ 2)}$. If $g \in G_0$ and $h \in G$ then $gh = hg$. But if $g, h \in G_1$ then $gh = -hg$.

For a given variety $\mathcal{V}$ of algebras, a superalgebra $A = A_0 + A_1$ is called a $\mathcal{V}$-*superalgebra* if its Grassmann envelope

$$G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$$

belongs to $\mathcal{V}$. It is easy to see that the Grassmann superalgebra $G = G_0 + G_1$ is commutative.

Every algebra A in a variety $\mathcal{V}$ can be imbedded into the $\mathcal{V}$-superalgebra $G \otimes A$ by means of the extension of the field of scalars F to the "domain of superscalars" G. Indeed, $G \otimes A$ inherits naturally $\mathbb{Z}_2$-grading of G: $(G \otimes A)_0 = G_0 \otimes A$, $(G \otimes A)_1 = G_1 \otimes A$, and is a $\mathcal{V}$-superalgebra, since $G(G \otimes A) = G(G) \otimes A$ and $G(G)$ is an associative commutative algebra.

Passage to superscalar extensions allow us to reduce the number of variables in the identities that are multilinear and either symmetric or skew-symmetric on certain variables.

It was Kemer [7] who first applied superalgebras to the investigation of varieties of associative algebras, in his solution of the famous Specht problem. Then this method was extended in the papers by Zel'manov [23] and Zel'manov-Shestakov [24] to investigation of nilpotence and solvability problems in non-associative algebras. Finally, Vaughan-Lee [22] applied superalgebras to reduce the number of variables for his computer calculations in the variety of associative nil-algebras of degree 4.

Notice that some times it is more convenient to define a $\mathcal{V}$-superalgebra by superidentities. To pass from $\mathcal{V}$-algebras to $\mathcal{V}$-superalgebras one has for any identity of $\mathcal{V}$ find an equivalent system of multilinear identities, and then apply to each multilinear identity so called "superization rule" (or "Kaplansky's principle") that whenever two odd variables are transposed a negative sign is introduced. For example, a commutative superalgebra is defined by

$$xy - (-1)^{\bar{x}\bar{y}} yx = 0,$$

and an alternative superalgebra is defined by

$$(x, y, z) + (-1)^{\bar{x}\bar{y}}(y, x, z) = 0 \qquad \text{(left super-alternativity)},$$
$$(x, y, z) + (-1)^{\bar{y}\bar{z}}(x, z, y) = 0 \qquad \text{(right super-alternativity)}.$$

Denote by

$$[x, y]_s = xy - (-1)^{\bar{x}\bar{y}} yx$$

the *super-commutator* of the homogeneous elements x, y, and by

$$x \circ_s y = xy + (-1)^{\bar{x}\bar{y}} yx$$

their *super-Jordan product.*

Let $\mathcal{V}[S; X]$ denote the free $\mathcal{V}$-superalgebra over a field F of characteristic zero generated by a set S of even generators and a set X of odd generators.

We use the following correspondence between the free $\mathcal{V}$-superalgebra $\mathcal{V}[\emptyset; x]$ on one odd generator x and the subspace of multilinear skew-symmetric elements $\mathcal{V}_{skew}[S; \emptyset]$ of the free $\mathcal{V}$-algebra $\mathcal{V}[S; \emptyset]$ on countable set of even generators $S = \{s_1, s_2, \ldots\}$.

Let $f = f(x)$ be a homogeneous non-associative polynomial of degree n on one variable x. It may be written in the form $f(x) = \tilde{f}(x, x, \ldots, x)$, for a certain multilinear polynomial $\tilde{f}(s_1, s_2, \ldots, s_n)$. Define the skew-symmetric polynomial $Skew\, f = Skew\, f\,(s_1, s_2, \ldots, s_n)$ as follows:

$$Skew\, f(s_1, s_2, \ldots, s_n) = \sum_{Alt} \tilde{f}(s_1, s_2, \ldots, s_n).$$

Then, $f(x) = 0$ in the superalgebra $\mathcal{V}[\emptyset; x]$ if and only if $Skew\, f\,(s_1, s_2, \ldots, s_n) = 0$ in the algebra $\mathcal{V}[S; \emptyset]$.

4 Free (super)algebras

In this section we formulate some open problems and state the main results of our research on free (super)algebras.

The construction of effective bases of free algebras is one of the most important and difficult problems in the theory of non-associative algebras. There are not many classes of algebras where such bases are known: free non-associative, free (anti)commutative and free Lie algebras are the most well-known examples besides polynomials and free associative algebras.

For every variety $\mathcal{V}$ of algebras, one can consider the corresponding *$\mathcal{V}$-Grassmann algebra* (see [17], [20]), which is isomorphic as a vector space to the subspace of all skew-symmetric elements of the free $\mathcal{V}$-algebra. Thus, it seems interesting to construct a base for this subspace. Due to the superalgebra method described above, the problem is reduced to the free $\mathcal{V}$-superalgebra on one odd generator, which is easier to deal with.

In order to obtain our results on skew-symmetric identities of an octonion algebra we first construct a base of the free quadratic alternative superalgebra generated by one odd element. It was done in several steps.

4.1 The free Malcev superalgebra on one odd generator

Let us denote by A^- the algebra obtained from an algebra A by replacing the product xy with the commutator $[x, y] = xy - yx$. Starting with an associative algebra A, one obtains in this way a Lie algebra A^-, and conversely, the celebrated Poincaré-Birkhoff-Witt Theorem establishes that every Lie algebra is isomorphic to a subalgebra of A^- for some associative algebra A. A weaker condition than the associativity for an algebra is the alternativity. For any alternative algebra A, the commutator algebra A^- is a Malcev algebra. However, at this time it remains an open problem whether any Malcev algebra is *special* [2], [16], i.e. isomorphic to a subalgebra of A^- for some alternative algebra A.

We recall that an anticommutative algebra is called a *Malcev algebra* if it satisfies the identity

$$J(x, y, z)x = J(x, y, xz),$$

where $J(x,y,z) = (xy)z + (yz)x + (zx)y$ is the jacobian of x, y and z [8], [10], [13]. Since for a Lie algebra the jacobian of any three elements vanishes, Lie algebras fall into the variety of Malcev algebras. Among the non-Lie Malcev algebras, the traceless elements of an octonion algebra with the product given by the commutator $[x,y] = xy - yx$ is one of the most important examples [8], [9], [13].

Notice that some methods used for associative and Lie algebras can be generalized for alternative and Malcev algebras. However, no effective bases are known for free alternative algebras or for free Malcev algebras.

In [16] Shestakov constructed a base of the free Malcev superalgebra $\mathcal{M} = Malc[\emptyset; x]$ on one odd generator: define inductively $x^{i+1} = x^i x$ then the elements

$$x^k,\ x^{4k}x^2,\ x^{4k+1}x^2,$$

where $k > 0$, form a base of $\mathcal{M}$. He also found an infinite family of skew-symmetric elements that are central in any Malcev or alternative algebra. At the time it was an open question whether the given family formed a base of all central skew-symmetric elements.

In [17] we constructed a base of the universal multiplicative envelope of $\mathcal{M}$. This allowed us to solve the question from [16] on a base of the space $Malc_{zskew}[S;\emptyset]$ of central skew-symmetric elements of the free Malcev algebra $Malc[S;\emptyset]$ of countable rank in characteristic zero: the elements

$$Skew\,(x^{4k}x^2)\,(s_{i_1}, s_{i_2}, \ldots, s_{i_{4k+2}}),\ k > 1,$$
$$Skew\,(x^{4k+1}x^2)\,(s_{i_1}, s_{i_2}, \ldots, s_{i_{4k+3}}),\ k > 0,$$

where $i_1 < i_2 < \cdots < i_{4k+3}$, form a base of the space $Malc_{zskew}[S;\emptyset]$.

In is known that Bol algebras generalize the notion of Malcev algebras. It would be interesting to investigate the free left Bol superalgebra on one odd generator in order to construct its base.

4.2 The free alternative superalgebra on one odd generator

The universal alternative envelope of $\mathcal{M}$ is isomorphic to the free alternative superalgebra $\mathcal{A} = Alt[\emptyset; x]$ generated by an odd element x with the universal homomorphism $\varphi : \mathcal{M} \to \mathcal{A}^-$ defined by $\varphi(x) = x$. In [18], [20] we used a quantization deformation of the Malcev Poisson superalgebra related with $\mathcal{M}$ according to [15] to construct a base of $\mathcal{A}$: define by induction

$$x^{[1]} = x,\ x^{[i+1]} = [x^{[i]}, x]_s,\ i > 0,$$

and denote

$$t = x^{[2]},\ z^{[k]} = [x^{[k]}, t],\ u^{[k]} = x^{[k]} \circ_s x^{[3]},\ k > 1,$$

then the elements

$$t^m x^\sigma,\ m + \sigma \geq 1, \quad t^m(x^{[k+2]}x^\sigma),$$
$$t^m(u^{[4k+\varepsilon]}x^\sigma), \quad t^m(z^{[4k+\varepsilon]}x^\sigma),$$

where $k > 0$, $m \geq 0$; ε, $\sigma \in \{0, 1\}$, form a base of the superalgebra $\mathcal{A}$.

The homomorphism φ maps the base of $\mathcal{M}$ to the linearly independent elements in $\mathcal{A}$, therefore, the superalgebra $\mathcal{M}$ is special. Notice that, in [19] we proved speciality of $\mathcal{M}$ directly, without knowing a base of $\mathcal{A}$.

In [17], [19] we proved that the elements

$$Skew\, z^{[k]}(s_1, \ldots, s_{k+2}),\ k \in \{4n, 4n+1\},\ k > 4,$$

are non-zero skew-symmetric central functions in $Alt\,[S; \emptyset]$. It would be interesting to find all skew-symmetric central and nuclear functions for alternative algebras. Evidently, they all should be of the type $Skew\, f$, where $f \in Z(\mathcal{A})$ and $f \in N(\mathcal{A})$ for central and nuclear functions, respectively. We recall that the nucleus $N(\mathcal{A})$ and the center $Z(\mathcal{A})$ are defined by

$$\begin{aligned} N(\mathcal{A}) &= \{a \in \mathcal{A} \mid (a, b, c) = (b, a, c) = (b, c, a) = 0, \text{ for any } b, c \in \mathcal{A}\}, \\ Z(\mathcal{A}) &= \{a \in N(\mathcal{A}) \mid [a, b]_s = 0, \text{ for any } b \in \mathcal{A}\}. \end{aligned}$$

In [20] we proved that

$$\begin{aligned} N(\mathcal{A}) &= id_{\mathcal{A}}\langle u^{[k]}, z^{[k]} \mid k > 2\rangle, \\ Z(\mathcal{A}) &= vect_F\langle t^m z^{[k]},\ t^m(2z^{[k]}x - u^{[k]}) \mid m \geq 0,\ k > 2\rangle. \end{aligned}$$

Notice that not every element in $Z(\mathcal{A})$ produces a central or nuclear function. For example, $z^{[4]} \in Z(\mathcal{A})$ but $Skew\, z^{[4]}$ is neither a central function nor a nuclear function in the algebra of octonions $\mathbb{O}$.

4.3 The free quadratic alternative superalgebra on one odd generator

We use a more general definition of a quadratic alternative algebra. For a unital algebra A, we identify F with the subalgebra $F \cdot 1$ of the algebra A.

We call a linear map τ of A into its center, $Z(A)$, a *trace* if, for any $x, y \in A$, it satisfies

$$\tau([x, y]) = \tau((x, y, z)) = 0,$$
$$\tau(\tau(x)y) = \tau(x)\tau(y).$$

We call a unital alternative algebra (A, τ) with a trace τ a *quadratic alternative* if it satisfies the linearized trace identity

$$x \circ y - \tau(x)y - \tau(y)x - \tau(xy) + \tau(x)\tau(y) = 0,$$

and $\tau(1) = 2$. In particular, every unital associative commutative algebra A is quadratic alternative in this sense if we put $\tau(a) = 2a$ for any $a \in A$.

The notion of a quadratic alternative algebra can be generalized to superalgebras. We consider the free quadratic alternative superalgebra $\mathcal{B}_\tau = (\mathcal{B}_\tau, \tau)$ on one odd generator x, that is, the free one-odd-generator object in the category of unital alternative superalgebras with supertrace τ that satisfies $\tau(1) = 2$ and

$$x \circ_s y - \tau(x)y - (-1)^{\bar{x}\bar{y}}\tau(y)x - \tau(xy) + \tau(x)\tau(y) = 0.$$

By $\mathcal{B}$ we denote the subsuperalgebra of $\mathcal{B}_\tau$, generated by x without using the supertrace operation.

In [21] we proved that the following elements

$$1, x, \tau(x), t, \tau(x)x, tx, \tau(x)t, \tau(tx), t^2, \tau(tx)x, \tau(x)\tau(tx), \tau(x)tx, \tau(tx)t,$$
$$\tau(x)\tau(tx)x, \tau(x)t^2, \tau(tx)tx, \tau(x)\tau(tx)t, \tau(tx)t^2, \tau(x)\tau(tx)tx, \tau(x)\tau(tx)t^2$$

form a base of the superalgebra $\mathcal{B}_\tau$. The subsuperalgebra $\mathcal{B}$ is nilpotent of degree 9, and the elements

$$x, tx^\sigma, t^2x^\sigma, x^{[3]}x^\sigma, x^{[4]}x^\sigma, x^{[5]}x^\sigma, z^{[4]}x^\sigma, u^{[4]}x^\sigma,$$

where $\sigma \in \{0, 1\}$, form a base of $\mathcal{B}$. Moreover, the T-ideal of super-identities on one odd generator of quadratic alternative superalgebras is generated by

$$tx^{[3]},\ z^{[4]} + x^{[6]},\ 12t^3 - x^{[6]}.$$

The T-subsuperalgebra of central functions on one odd generator of quadratic alternative superalgebras is generated by

$$t^2,\ 12t^2x - x^{[5]}.$$

Now, the images of these elements under the map *Skew* described in Section 3 give us skew-symmetric identities and central functions of a quadratic alternative algebra in characteristic zero. In order to obtain the Theorem from Introduction we proved that the skew-symmetric identities and central functions of an octonion algebra coincides with those for the class of all quadratic alternative algebras.

We conjecture that the T-ideal of identities of quadratic alternative algebras coincides with the identities of octonions. As supporting evidence for this conjecture we mention the case of associative algebras where the quadratic identity implies all identities of quaternions. This is a partial case of the Razmyslov theorem [12] which states that over a field of characteristic zero the Cayley-Hamilton identity implies all identities of $n \times n$ matrices.

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