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Nestandardní číselné soustavy

Non-standard numeration systems

Summary

We consider representations of numbers by infinite iteration of a positive function f, introduced by Rényi. For a special choice of the function f we obtain numeration systems related to a base $\beta > 1$, which for $\beta \in \mathbb{Z}$ corresponds to usual β-adic numeration.

Every non-negative x has a unique expression in the form of its β-expansion $x = \sum_{i=-\infty}^{k} x_i \beta^i$. As an analogy to ordinary rational integers $\mathbb{Z}$, one defines the set $\mathbb{Z}_\beta$ of β-integers, i.e. numbers whose β-expansion is of the form $x = \sum_{i=0}^{k} x_i \beta^i$. If β is an integer, then $\mathbb{Z}_\beta = \mathbb{Z}$. For $\beta \notin \mathbb{Z}$, the set $\mathbb{Z}_\beta$ has certain peculiar properties. For example, not every finite sequence $x_k \dots x_0$ of coefficients $x_i \in \{0, 1, \dots, [\beta]\}$ is a β-expansion of some $x > 0$. The admissibility of sequences of coefficients is given by the Parry condition using the so-called Rényi expansion $d_\beta(1)$ of 1.

The set $\mathbb{Z}_\beta$ for $\beta \notin \mathbb{Z}$ displays arithmetical behaviour different from that for integer bases of numeration. Not only $\mathbb{Z}_\beta$ is not closed under addition and multiplication, but the result of summing and multiplying β-integers may even have an infinite β-expansion. This means that the set $\mathrm{Fin}(\beta)$ of finite β-expansions is not necessarily a ring. We list several sufficient conditions so that $\mathrm{Fin}(\beta)$ is closed under addition and multiplication. One can also study the bounds on the length of the appearing β-fractional part. Estimates of these bounds are known only for certain classes of bases β.

Souhrn

Uvažujeme reprezentace čísel pomocí nekonečných iterací pozitivní funkce f, jak je zavedl A. Rényi. Pro vhodnou volbu funkce f získáme číselnou soustavu s bází $\beta > 1$, která pro celočíselné β odpovídá obvyklé β-adické číselné soustavě.

Každé nezáporné číslo x má jednoznačné vyjádření ve tvaru β-rozvoje $x = \sum_{i=-\infty}^{k} x_i \beta^i$. Jako analogii k racionálním celým číslům $\mathbb{Z}$, zavádíme množinu $\mathbb{Z}_\beta$ β-celých čísel, tj. čísel, jejichž β-rozvoj je tvaru $x = \sum_{i=0}^{k} x_i \beta^i$. Pokud β je celé číslo, je $\mathbb{Z}_\beta = \mathbb{Z}$. Jestliže $\beta \notin \mathbb{Z}$, pak má množina $\mathbb{Z}_\beta$ některé neobvyklé vlastnosti. Například ne každá konečná posloupnost $x_k \dots x_0$ cifer $x_i \in \{0, 1, \dots, [\beta]\}$ je β-rozvojem nějakého $x > 0$. Přípustnost posloupností cifer je dána Parryho podmínkou, podle tzv. Rényiho rozvoje jedničky $d_\beta(1)$.

Množina $\mathbb{Z}_\beta$ má pro $\beta \notin \mathbb{Z}$ zvláštní chování vzhledem k aritmetickým operacím. Nejenže $\mathbb{Z}_\beta$ není uzavřená na sčítání a násobení, ale výsledek součtu nebo součinu β-celých čísel může dokonce být číslo s nekonečným β-rozvojem. To znamená, že množina $\mathrm{Fin}(\beta)$ konečných β-rozvojů není nutně okruh. Dáváme přehled několika postačujících podmínek na to, aby $\mathrm{Fin}(\beta)$ byla uzavřena na sčítání a násobení. Lze také studovat meze na délku β-zlomkové části, která při aritmetických operacích vzniká. Odhady těchto mezí jsou známé jen pro úzkou třídu bází β.

Klíčová slova: číselné soustavy, hladový algoritmus, β-rozvoj, β-celá čísla, Rényiův rozvoj jedničky, Pisotovo číslo, Parryho číslo, vlastnost konečnosti, Rauzyho fraktál, řetězové zlomky

Key words: numeration system, greedy algorithm, β-expansion, β-integers, Rényi expansion of 1, Pisot number, Parry number, finiteness property, Rauzy fractal, continued fraction

Contents

1 Introduction

Efficient manipulation of real numbers in computers is still a challenge. Many interesting theoretical and algorithmic problems are linked with that topic and belong to quite distant fields such as computer science, number theory, numerical analysis, computer algebra and logics. The crucial point is the study of various definitions and properties of numeration systems. Numbers are usually represented as finite or infinite words over a finite alphabet of digits. In usual positional numeration systems with an integer base q (q-adic systems) the representation of a number is found by the 'greedy algorithm'. Such systems share the properties of the most commonly used decimal ($q = 10$) or binary ($q = 2$) systems.

Rényi in his paper [18] has introduced a general framework, in which the usual q-adic systems are found as a very special case. He defines representations of a real number x by infinite iteration of a positive function f. According to the choice of the function f, one may obtain expressions of the number x in various forms, for example in the form of its continued fraction. Using another function f, with a parameter $\beta > 1$, one obtains the so-called β-expansion, which generalizes the ordinary q-adic systems for non-integral bases β.

If β is a rational integer greater than 1, then properties of the representation of x by its β-expansion are practically the same as for the decimal expansion of x. Numeration systems based on $\beta \notin \mathbb{Z}$ are essentially different, but are not less interesting, especially if β is an algebraic irrational number, (more specifically a Pisot number).

There are two main issues in the study of arithmetics on β-expansions. The first of them is the knowledge whether the arithmetical operations with finite β-expansion produce always results with finite β-fractional part. In other words, whether the set $\mathrm{Fin}(\beta)$ is closed under addition and multiplication, i.e. is a ring. A base β, which satisfies this, is said to have the finiteness property. Algebraic description of numbers with the finiteness property is an unsolved problem. An overview of results on this topic is given in Section 4.1. The second issue is the study of the β-fractional part that possibly appears in arithmetic operations with β-expansions. Section 4.2 is devoted to this problem.

2 Rényi representation of real numbers

In his paper [18], Rényi has introduced representations of a real number x by infinite iteration of a positive function $y = f(x)$ in the form of the f-expansion

$$x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + f(\varepsilon_3 + \cdots)\cdots)\,. \tag{1}$$

In the above expression, the 'digits' $\varepsilon_n = \varepsilon_n(x)$, $n \in \mathbb{N}_0$ and the 'remainders'

$$r_n(x) = f(\varepsilon_{n+1} + f(\varepsilon_{n+2} + f(\varepsilon_{n+3} + \cdots)\cdots)\,, \quad n \in \mathbb{N}_0\,,$$

are defined by the recursive relations

$$\begin{array}{rclcrcl} \varepsilon_0(x) &=& [x]\,, & & r_0(x) &=& \{x\}\,, \\ \varepsilon_{n+1}(x) &=& \big[\varphi(r_n(x))\big]\,, & & r_{n+1}(x) &=& \big\{\varphi(r_n(x))\big\}\,, \quad n \in \mathbb{N}_0\,, \end{array}$$

where φ is the inverse function of f. The notation $[z]$ stands for the integral part of the real number z, and $\{z\} = z - [z]$ is the fractional part of z.

Rényi further provides sufficient conditions on the function f, so that the representation (1) exists for every real x. One distinguishes two cases, according to whether the function f is decreasing or increasing.

Decreasing f: Let the function f satisfy the following conditions.

- $f(1) = 1$;
- $f(t)$ is a positive, continuous, strictly decreasing function for $1 \le t \le T$, and $f(t) = 0$ for $t \ge T$, where $2 < T \le +\infty$;
- $|f(t_2) - f(t_1)| \le |t_2 - t_1|$ for $1 \le t_1 < t_2$, and moreover, there exists $\lambda \in (0, 1)$ such that $|f(t_2) - f(t_1)| \le \lambda |t_2 - t_1|$ for $1 + f(2) < t_1 < t_2$.

Then the f-expansion (1) exists for every real number x. Dependently on the parameter T, the f-expansion of x may have different digits.

- If $T = +\infty$, then the digits $\varepsilon_n(x)$ may be all positive integers, $\varepsilon_n(x) \in \mathbb{N}$;
- If T is an integer, $2 < T < +\infty$, then $\varepsilon_n(x) \in \{1, 2, 3, \cdots, T - 1\}$;
- If $T \notin \mathbb{Z}$, $2 < T < +\infty$, then $\varepsilon_n(x)$ take values $\varepsilon_n(x) \in \{1, 2, 3, \cdots, [T]\}$.

Increasing f: Let the function f satisfy the following conditions.

- $f(0) = 0$;
- $f(t)$ is a continuous, strictly increasing function for $0 \le t \le T$, and $f(t) = 1$ for $t \ge T$, where $1 < T \le +\infty$;
- $|f(t_2) - f(t_1)| < |t_2 - t_1|$ for $0 \le t_1 < t_2$.

Then the f-expansion (1) exists for every real number x. Again, the parameter T decides about the digits of the f-expansion of x.

- If $T = +\infty$, then the digits $\varepsilon_n(x)$ may be all non-negative integers, $\varepsilon_n(x) \in \mathbb{N}_0$;
- If T is an integer, $2 < T < +\infty$, then $\varepsilon_n(x) \in \{0, 1, 2, 3, \cdots, T - 1\}$;
- If $T \notin \mathbb{Z}$, $2 < T < +\infty$, then $\varepsilon_n(x)$ take values $\varepsilon_n(x) \in \{0, 1, 2, 3, \cdots, [T]\}$.

This quite general notion of f-expansion of a real number x is a frame including the known expressions of x in the form of its continued fraction, or its q-adic expansion, where $q \in \mathbb{Z}$, $q \ge 2$. Let us show the functions f that lead to these.

2.1 Continued fractions

The representation of a real number z in the form of its continued fraction can be obtained by the Rényi algorithm, choosing for f the decreasing function

$$f(x) = \frac{1}{x}, \qquad \text{for } x \ge 1 .$$

Every positive real number z has a representation in the form

$$z = \varepsilon_0 + \cfrac{1}{\varepsilon_1 + \cfrac{1}{\varepsilon_2 + \cfrac{1}{\varepsilon_3 + \cdots}}} .$$

Since the function inverse to f is $\varphi(y) = \frac{1}{y}$, the digits ε_n, $n \in \mathbb{N}_0$, are calculated by

$$\begin{array}{rclcrcl} \varepsilon_0 &=& [z]\,, & \qquad & r_0 &=& \{z\}\,, \\ \varepsilon_1 &=& [\varphi(r_0)] = \left[\frac{1}{\{z\}}\right]\,, & & r_1 &=& \left\{\frac{1}{\{z\}}\right\}\,, \\ &\vdots& & & &\vdots& \end{array}$$

which is the usual algorithm for calculating coefficients of the continued fraction of z. It is not difficult to verify that the function $f(x) = \frac{1}{x}$ has all required properties (in the case of a decreasing function), the parameter $T = +\infty$, and thus the digits ε_n, $n \in \mathbb{N}$, may take any values among positive integers.

Example 1. As an example, let us show the development into continued fraction of the irrational number $\tau = \frac{1}{2}(1+\sqrt{5}) = 1.618\cdots$, the so-called golden ratio. The coefficients $\varepsilon_n(\tau)$ are calculated by

$$\begin{array}{rclcrcl} \varepsilon_0 &=& [\tau] = [1.618\cdots] = 1\,, & \qquad & r_0 &=& \tau - 1 = \frac{1}{\tau}\,, \\ \varepsilon_1 &=& \left[\frac{1}{\tau-1}\right] = [\tau] = 1\,, & & r_1 &=& \tau - 1 = r_0\,. \end{array}$$

We have used the fact that τ satisfies $\tau^2 = \tau + 1$. Since $r_1 = r_0$, it is obvious, that $\varepsilon_n(\tau) = 1$ for all $n \in \mathbb{N}_0$, and thus the continued fraction of the golden ratio is

$$\tau = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}\,.$$

2.2 q-adic expansions

Let

$$f(x) = \begin{cases} \dfrac{x}{q}\,, & \text{for } 0 \le x \le q\,, \\ 1\,, & \text{for } q < x\,, \end{cases} \tag{2}$$

where q is an integer $q \ge 2$. Every positive real number z has a representation in the form

$$z = \varepsilon_0 + \frac{\varepsilon_1}{q} + \frac{\varepsilon_2}{q^2} + \frac{\varepsilon_3}{q^3} + \cdots\,,$$

where the digits ε_n, $n \in \mathbb{N}_0$, are calculated using the inverse function $\varphi(y) = qy$ by

$$\begin{array}{rclcrcl} \varepsilon_0 &=& [z]\,, & \qquad & r_0 &=& \{z\}\,, \\ \varepsilon_1 &=& \big[q\{z\}\big]\,, & & r_1 &=& \big\{q\{z\}\big\}\,, \\ &\vdots& & & &\vdots& \end{array}$$

The function f verifies the conditions for increasing functions with $T = q \in \mathbb{Z}$. The digits take values $\varepsilon_n \in \{0, 1, 2, \ldots, q-1\}$.

Example 2. When $q = 10$, the f-expansion corresponds to the ordinary decimal expansion of real numbers. For example the digits of $z = \pi = 3.14159\cdots$ are calculated by

$$\begin{array}{rclcrcl} \varepsilon_0(\pi) &=& [\pi] = 3\,, & \qquad & r_0(\pi) &=& \pi - 3\,, \\ \varepsilon_1(\pi) &=& \big[10(\pi - 3)\big] = [1.4159\cdots] = 1\,, & & &\vdots& \end{array}$$

3 β-expansions

In the previous section we have illustrated the Rényi construction of f-expansions on two examples, continued fractions and q-adic expansions for $q \in \mathbb{Z}$. In the first case, the parameter of the function f satisfied $T = +\infty$, in the second, $T \in \mathbb{Z}$. As Rényi has shown, in such cases (i.e. when T is not a non-integral real number), every finite sequence of admissible digits $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ corresponds to an f-expansion of some real number x. On the other hand, if $T < +\infty$, but $T \notin \mathbb{Z}$, then this is not true. This phenomena will be well illustrated on the notion of β-expansions.

The β-expansion of x is obtained if for the Rényi algorithm of f-expansion we use the function f of the form (2), where instead of the integer parameter q we use any real number $\beta > 1$. Let us study the so-called β-numeration in more detail.

Let $\beta > 1$. Any positive x has a representation in the form of a series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i = x_k \beta^k + \cdots + x_1 \beta + x_0 + \frac{x_{-1}}{\beta} + \frac{x_{-2}}{\beta^2} + \cdots , \qquad x_i \in \mathbb{N}_0 ,$$

which can be formally written as

$$(x)_\beta = x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots ,$$

i.e. the number x is represented by a word over the alphabet of non-negative integers, and the symbol $\bullet$ marks the separation between digits with non-negative and with negative indexes, (β-integer and β-fractional part).

Such representation is in general not unique. For example, if $\beta = 10$, than for $x = 1$ we have

$$x = 1 \cdot 10^0 + \frac{0}{10} + \frac{0}{10^2} + \cdots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots ,$$

i.e. x is represented by the word $1 \bullet 000 \cdots$, but also by the word $0 \bullet 999 \cdots$

The β-expansion of a given $x > 0$ is the representation of x whose digits are obtained in the following way: We define the transformation $T_\beta : [0,1] \to [0,1)$, by

$$T_\beta(x) := \beta x - [\beta x] = \{\beta x\} .$$

- If $x \in [0,1)$, then put $x_i = \left[\beta T_\beta^{-i-1}(x)\right]$ for $i = -1, -2, -3, \cdots$.
- If $x > 1$, then find the maximal exponent k for which β^k is smaller or equal to x, i.e. such that $\beta^k \le x < \beta^{k+1}$. Then $\frac{x}{\beta^{k+1}} \in [0,1)$ and we can put $x_i = \left[\beta T_\beta^{k-i}\left(\frac{x}{\beta^{k+1}}\right)\right]$ for $i = k, k-1, k-2, \cdots$.
- If $x = 1$, put $x_0 = 1$ and $x_i = 0$ for $i < 0$.

Clearly, for $x \in [0,1)$ this definition corresponds to the Rényi f-expansion of x with f given above. Note that the β-expansion of $x = 1$ is $(x)_\beta = 1 \bullet 000 \cdots$, which is different from the sequence of digits obtained by applying the transformation T_β. We define the so-called Rényi expansion $d_\beta(1)$ of 1, as follows.

$$d_\beta(1) = t_1 t_2 t_3 \cdots , \qquad \text{where} \quad t_i = \left[\beta T_\beta^{i-1}(1)\right] .$$

We have

$$1 = \sum_{i=1}^{+\infty} \frac{t_i}{\beta^i} .$$

The Rényi expansion $d_\beta(1)$ of 1 is a crucial element in describing the properties of the numeration system with the base β.

3.1 The Parry condition

As we have said, for $\beta \in \mathbb{Z}$, all finite sequences of admissible digits, i.e. in $\{0, 1, \ldots, \beta-1\}$, correspond to β-expansions of some $x > 0$. Among the infinite sequences of these digits, only those are forbidden, which end in $(\beta - 1)^\omega$, as seen on the example of decimal numeration system, where the expansion of no real number ends with an infinite sequence of digits 9. If the base β is not a rational integer, i.e $\beta \notin \mathbb{Z}$, the situation is more complicated. In order to decide which sequences of digits correspond to a β-expansion, one uses the Parry condition [17], which is based on the Rényi expansion $d_\beta(1)$ of 1.

If among the coefficients t_i of $d_\beta(1)$ there are infinitely many non zeros, we put $d^*_\beta(1) = d_\beta(1)$. On the other hand, if m is the maximal index of a non-zero coefficient of $d_\beta(1)$, i.e.

$$1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \cdots + \frac{t_m}{\beta^m}\,, \qquad t_m \neq 0\,,$$

then we put $d^*_\beta(1) = \big(t_1 t_2 \ldots t_{m-1}(t_m - 1)\big)\big(t_1 t_2 \ldots t_{m-1}(t_m - 1)\big)\ldots$. The sequence $d^*_\beta(1)$ is eventually periodic with period of length m.

Theorem 1 (Parry). *Then series $\sum_{i=-\infty}^{k} x_i \beta^i$, $x_i \in \mathbb{N}_0$, is the β-expansion of some real number $x > 0$ if and only if for all $j \leq k$, the sequence $x_j x_{j-1} x_{j-2} \ldots$ is lexicographically strictly smaller than the sequence $d^*_\beta(1)$. Formally,*

$$x_j x_{j-1} x_{j-2} \cdots \prec d^*_\beta(1)\,.$$

Example 3. Let us illustrate the Parry condition on the example of the numeration system with the base $\beta = \tau = \frac{1}{2}(1 + \sqrt{5})$. The digits of τ-expansions are 0 and $1 = [\tau]$. Since

$$\begin{aligned}
T_\tau^0(1) &= 1\\
T_\tau^1(1) &= \tau - [\tau] = \tau - 1 = \tau^{-1}\\
T_\tau^2(1) &= T_\tau(\tau^{-1}) = \tau\tau^{-1} - [\tau\tau^{-1}] = 0\\
T_\tau^k(1) &= 0 \qquad \text{for } \ k \geq 3\,,
\end{aligned}$$

the Rényi expansion $d_\tau(1)$ is calculated as follows.

$$\begin{aligned}
t_1 &= [\tau T_\tau^0(1)] = [\tau] = 1\\
t_2 &= [\tau T_\tau^1(1)] = [\tau\tau^{-1}] = 1\\
t_k &= [\tau T_\tau^k(1)] = [0] = 0 \qquad \text{for } \ k \geq 3\,.
\end{aligned}$$

Thus $d_\tau(1) = 11$, which corresponds to the fact that $1 = \frac{1}{\tau} + \frac{1}{\tau^2}$. Then $d^*_\tau(1) = (10)^\omega$. The Parry condition tells that a sequence $x_k x_{k-1} \cdots x_0 \bullet x_{-1} x_{-2}$ is a τ-expansion, if $x_j x_{j-1} x_{j-2} \cdots$ is lexicographically smaller than $1010(10)^\omega$ for all $j \leq k$. This means that a τ-expansion has only digits 1 and 0, does not contain the string 11, and does not end with the period $(10)^\omega$.

3.2 Parry number, Pisot number

Among the bases of numeration systems, those numbers $\beta > 1$ are important for which the Rényi expansion $d_\beta(1)$ is eventually periodic. Such numbers β are called *beta-numbers* [17] or *Parry numbers*. It is not difficult to see that a Parry number β is a root of an equation

$$x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0 = 0\,, \qquad a_{n-1}, \ldots, a_1, a_0 \in \mathbb{Z}\,. \tag{3}$$

Consequently, β is an algebraic integer. We denote its conjugates as $\beta^{(2)}, \ldots, \beta^{(d)}$, where d is the degree of β. If we consider only one conjugate of β, we use for it the simple notation β'. The corresponding isomorphisms between the fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta^{(i)})$ (or $\mathbb{Q}(\beta')$) associate $x \mapsto x^{(i)}$, x' respectively.

It is known that all Parry numbers are Perron numbers, i.e. algebraic integers > 1, whose all conjugates are smaller than β in modulus. On the other hand, among the Parry numbers there are Pisot numbers, i.e. algebraic integers > 1, whose all conjugates are smaller than 1 in modulus, see [8], [19].

Example 4. The golden ratio $\tau = \frac{1}{2}(1+\sqrt{5})$, with $d_\tau(1) = 11$, is a Pisot number, since it satisfies the quadratic equation $x^2 = x + 1$ and its conjugate $\tau' = \frac{1}{2}(1-\sqrt{5})$ belongs to $(-1, 0)$. On the other hand, the number $\beta = \frac{3}{2}$ is not even a Perron number, since it is not an algebraic integer. Its Rényi expansion $d_\beta(1)$ is infinite non-periodic.

A complete algebraic description of Parry numbers is an open question, but certain results can be found in works [9], [10], and [20].

3.3 β-integers and their geometric properties

The set of numbers with eventually periodic β-expansion is denoted by $\mathrm{Per}(\beta)$. The set of all numbers x with finite β-expansion, i.e. such that $(|x|)_\beta$ ends in infinitely many zeros, is denoted

$$\mathrm{Fin}(\beta) := \{x \in \mathbb{R} \mid (|x|)_\beta = x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots x_\ell\}.$$

The real numbers x, such that the β-expansion of $|x|$ has vanishing coefficients at negative powers of β are called β-integers. The set of β-integers is denoted by

$$\mathbb{Z}_\beta := \{x \in \mathbb{R} \mid (|x|)_\beta = x_k x_{k-1} \ldots x_0\}.$$

If β is a rational integer, then the set of β-integers coincides with $\mathbb{Z}$. It means that it is a ring, i.e. closed under addition and multiplication. If drawn on the real line, the β-integers are equidistant with the distance between adjacent elements being 1. Also the set $\mathrm{Fin}(\beta)$ of finite β-expansions has the ring structure. The set of eventually periodic β-expansions is the field of rational numbers, $\mathrm{Per}(\beta) = \mathbb{Q}$.

The situation is much more complicated if $\beta \notin \mathbb{Z}$. The structure of $\mathbb{Z}_\beta$ is interesting if β is a Parry number, i.e. such that the Rényi expansion $d_\beta(1)$ is eventually periodic. In that case, the values of distances between adjacent β-integers are finitely many.

Determining the distances between adjacent β-integers is not difficult, if we realize that lexicographical ordering on β-expansions corresponds to the natural ordering of numbers. More precisely, if $x > y > 0$ are numbers with β-expansions

$$(x)_\beta = x_k x_{k-1} \cdots x_0, \qquad (y)_\beta = y_l y_{l-1} \cdots x_0,$$

then $k \geq l$ and

$$x_k x_{k-1} \cdots x_0 \succ \underbrace{00\cdots0}_{(l-k)\text{times}} y_l y_{l-1} \cdots y_0.$$

The opposite is also true. Therefore $x < y$ are adjacent β-integers, if their β-expansions are of the form

$$\begin{aligned}(x)_\beta &= x_k\ x_{k-1}\ \cdots\ x_{i+1}\quad x_i\quad t_1\ t_2\ \cdots\ t_i,\\ (y)_\beta &= x_k\ x_{k-1}\ \cdots\ x_{i+1}(x_i+1)\,0\ \ 0\ \cdots\ 0.\end{aligned}$$

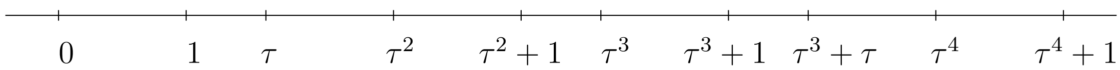
Thus the distance between x and y is

$$y - x = \begin{array}{r} 1\ 0\ 0\ \cdots\ 0 \phantom{\bullet\ t_{i+1}\ t_{i+2}\ \cdots} \\ -\ t_1\ t_2\ \cdots\ t_i \phantom{\bullet\ t_{i+1}\ t_{i+2}\ \cdots} \\ \hline t_1\ t_2\ \cdots\ t_i \bullet t_{i+1}\ t_{i+2}\ \cdots \\ -\ t_1\ t_2\ \cdots\ t_i \phantom{\bullet\ t_{i+1}\ t_{i+2}\ \cdots} \\ \hline 0 \bullet t_{i+1}\ t_{i+2}\ \cdots \end{array}$$

The possible distances are thus obtained as $0 \bullet t_{i+1}t_{i+2}\cdots$ for $i \geq 0$. Obviously, if the Rényi expansion $d_\beta(1)$ is eventually periodic, which is the case of Parry numbers β, then there is only finite number of possible distances between adjacent β-integers.

Example 5. If β is the golden ratio τ, then the distances between adjacent τ-integers are $\frac{1}{\tau} + \frac{1}{\tau^2} = 1$ corresponding to the expansion $0 \bullet 11$, and $\frac{1}{\tau}$, corresponding to $0 \bullet 1$. Let us list several smallest elements of $\mathbb{Z}_\tau \cap [0, +\infty)$, together with their τ-expansion,

x	$(x)_\tau$
0	0
1	1
τ	10
τ^2	100
τ^2+1	101
τ^3	1000
τ^3+1	1001
$\tau^3+\tau$	1010
τ^4	10000
τ^4+1	10001
$\vdots$	$\vdots$



An important tool in the study of numeration systems is the notion of the central tile, or Rauzy fractal. Let us explain it on the example of the Tribonacci numeration system.

Example 6. The Tribonacci number β is a Pisot number, solution to $x^3 = x^2+x+1$, whose conjugates are mutually conjugated complex numbers β' and $\beta'' = \bar{\beta}'$, in modulus smaller than 1. The central tile is the closure of the image of $\mathbb{Z}_\beta$ under the field isomorphism $' : \mathbb{Q}(\beta) \mapsto \mathbb{Q}(\beta')$. The central tile is a bounded set in the complex plane, since β is a Pisot number, i.e. $|\beta'| < 1$, and thus images of all β-integers are bounded,

$$|x'| < \sum_{i=0}^{\infty} |\beta'|^i = \frac{1}{1 - |\beta'|}\,.$$

Figure 1 shows the set

$$\mathbb{Z}'_\beta = \left\{ \sum_{i=0}^{k} a x_i \beta'^i \;\middle|\; \sum_{i=0}^{k} x_i \beta^i \in \mathbb{Z}_\beta \right\}$$

drawn in the complex plane. Bernat in [7] has shown that the closure of $\mathbb{Z}'_\beta$ is a set centrally symmetric with respect to the point $c = \frac{1}{2}(1 - \beta')^{-1}$.

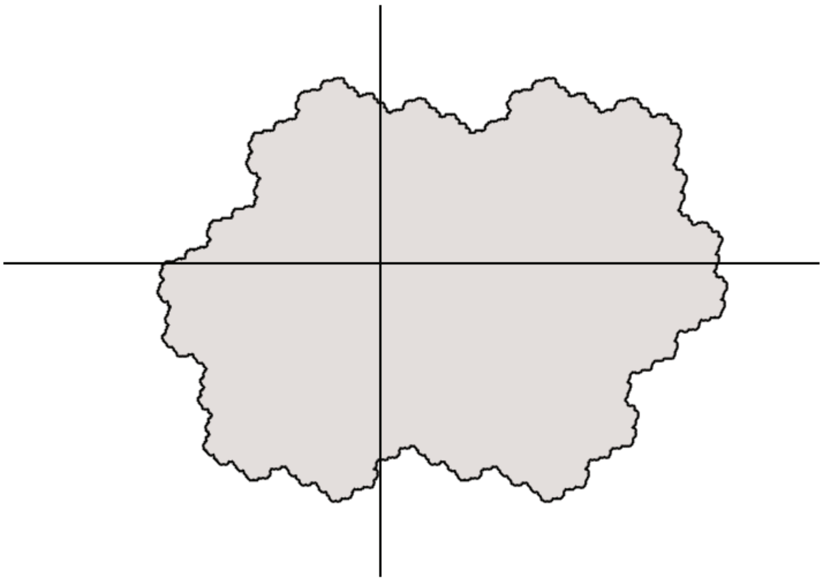


Figure 1: Rauzy fractal of the Tribonacci numeration system.

4 Arithmetics on β-expansions

Let us compare the arithmetic properties of β-expansions for non-integral β with the arithmetics in usual numeration systems. If the base of the numeration satisfies $\beta \in \mathbb{Z}$, we have $\mathbb{Z}_\beta = \mathbb{Z}$, thus it is a ring, and $\mathrm{Per}(\beta)$ equals to the field $\mathbb{Q}$ of rational numbers.

If the base β of the numeration system is not an integer, the situation is much more complicated. First of all, the set $\mathbb{Z}_\beta$ of β-integers is not closed under addition and multiplication. For example, in the numeration system based on the golden ratio τ, we have $1 + 1 = 2 = \tau + \frac{1}{\tau^2}$, i.e. the τ-expansion of 2 is $(2)_\tau = 10 \bullet 01$.

In fact, very little is known about arithmetical properties of β-expansions in general. Among the non-integral bases of numeration, Pisot numbers are of particular interest, and they are also the most studied. Schmidt [19] has shown that if β is a Pisot number, then $\mathrm{Per}(\beta)$ equals to the extension field $\mathbb{Q}(\beta)$, which is the analogy of the case of usual numeration systems.

4.1 Finiteness property

One of the questions about arithmetics on β-expansions is the description of such bases of numeration, which satisfy the so-called finiteness property, i.e. such that the set $\mathrm{Fin}(\beta)$ of finite β-expansions is closed under addition and multiplication.

It is not difficult to show that if the finiteness property is satisfied, then the Rényi expansion $d_\beta(1)$ of 1 cannot be infinite. For, we have

$$1 - \frac{1}{\beta} = \sum_{i=1}^{\infty} \frac{t_i}{\beta^i} - \frac{1}{\beta} = \frac{t_1 - 1}{\beta} + \sum_{i=2}^{\infty} \frac{t_i}{\beta^i}\,,$$

thus the β-expansion of the sum $x + y$, where $x = 1$ and $y = -\beta^{-1}$ is of the form $0 \bullet (t_1 - 1)t_2t_3\cdots$, which is finite only if $d_\beta(1)$ is finite.

First results about the finiteness property are due to Frougny and Solomyak. In their paper [12] they show that if β satisfies the finiteness property, then β is a Pisot number.

They provide also a sufficient condition on β, so that $\mathrm{Fin}(\beta)$ is a ring. They prove that if β is such that $d_\beta(1) = t_1 t_2 \cdots t_m$, where $t_1 \geq t_2 \geq \cdots \geq t_m$, then β has the finiteness property. Another sufficient condition was given by Hollander [14]. He has shown that $t_1 > t_2 + t_3 + \cdots t_m$ is a sufficient condition so that $\mathrm{Fin}(\beta)$ is a ring. For cubic unitary Pisot numbers β Akiyama [2] has shown that $\mathrm{Fin}(\beta)$ is a ring if and only if the Rényi expansion $d_\beta(1)$ is finite.

Another approach to the study of finiteness property of bases β of numeration is to study the images of β-integers under the isomorphisms of the fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta^{(i)})$. If β is a Pisot number, then they are bounded. We denote

$$H^{(i)} := \sup\{|x^{(i)}| \mid x \in \mathbb{Z}_\beta\} < +\infty\,,$$

for $i = 2, \ldots, d$, where d is the degree of β, $\beta^{(i)}$ is the i-th conjugate of β and $x^{(i)}$ is the image of x under the i-th isomorphism. In [13] it is shown that there exist finite sets $F_\oplus$ and $F_\otimes$ such that

$$\begin{array}{rcl} \mathbb{Z}_\beta + \mathbb{Z}_\beta & \subset & \mathbb{Z}_\beta + F_\oplus\,, \\ \mathbb{Z}_\beta \times \mathbb{Z}_\beta & \subset & \mathbb{Z}_\beta + F_\otimes\,, \end{array} \tag{4}$$

In [3] one studies the question of minimizing the finite sets $F_\oplus$ and $F_\otimes$, using their dependence on $H^{(i)}$, $i = 2, \ldots, d$. The β-expansions of elements of $F_\oplus$ and $F_\otimes$ play the role of β-fractional parts of sums and products of two β-integers. Thus if all elements of $F_\oplus$, $F_\otimes$ have finite β-expansion, then $\mathrm{Fin}(\beta)$ is a ring. It is however difficult to determine values $H^{(i)}$, and thus also finding the sets $F_\oplus$ and $F_\otimes$, in general. The algebraic description of bases β of numeration systems satisfying the finiteness property therefore remains an open question.

4.2 Fractional part arising in arithmetic operations

Even in case that $\mathrm{Fin}(\beta)$ is not closed under addition and multiplication, interesting information about the arithmetics in the β-numeration systems is given by the quantities

$$\begin{array}{rcl} L_\oplus = L_\oplus(\beta) & := & \min\{n \in \mathbb{N}_0 \mid \forall\, x, y \in \mathbb{Z}_\beta,\ x + y \in \mathrm{Fin}(\beta) \Rightarrow x + y \in \beta^{-n}\mathbb{Z}_\beta\}\,, \\ L_\otimes = L_\otimes(\beta) & := & \min\{n \in \mathbb{N}_0 \mid \forall\, x, y \in \mathbb{Z}_\beta,\ xy \in \mathrm{Fin}(\beta) \Rightarrow xy \in \beta^{-n}\mathbb{Z}_\beta\}\,. \end{array}$$

They are the bounds on the length of the fractional parts arising in addition and multiplication of β-integers. It is known [12] that $L_\oplus$, $L_\otimes$ are finite, if the base β of numeration is a Pisot number, whether $\mathrm{Fin}(\beta)$ is a ring or not.

One of the possibilities for providing some estimates on $L_\oplus$, $L_\otimes$, for β Pisot, is to find the β-expansion of all elements in the sets $F_\oplus$, $F_\otimes$ from (4). Very useful is the following theorem, which is applicable even if β is not a Pisot number.

Theorem 2 ([13]). *Let β be an algebraic number, $\beta > 1$, with at least one conjugate β' satisfying*

$$H := \sup\{|x'| \mid x \in \mathbb{Z}_\beta\} < +\infty\,, \qquad K := \inf\{|x'| \mid x \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta\} > 0\,,$$

where x' stands for the image of x under the isomorphism of fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta')$. Then

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus} < \frac{2H}{K} \qquad \text{and} \qquad \left(\frac{1}{|\beta'|}\right)^{L_\otimes} < \frac{H^2}{K}\,.$$

In the above theorem one requires existence of at least one conjugate of β such that the constant H is finite and K is positive. The former is ensured if $|\beta'| < 1$. Verifying whether $K > 0$ or $K = 0$ is much more complicated. Akiyama in [1] shows that, if β is a unit and $\mathrm{Fin}(\beta)$ is a ring, the origin is an inner point of the central tile in the conjugated plane. As a consequence, K is positive for all conjugates of β. This is a direct example of how Rauzy fractals are useful in the study of arithmetic properties of non-standard numeration systems.

Let us mention that values of $L_\oplus(\beta)$, $L_\otimes(\beta)$ are known for very few bases β of numeration. First known values have been provided in [11] for quadratic Pisot units. It is shown there that $L_\oplus(\beta) = L_\otimes(\beta) = 2$, if $\beta^2 = m\beta + 1$ for $m \in \mathbb{N}$, and $L_\oplus(\beta) = L_\otimes(\beta) = 1$, if $\beta^2 = m\beta - 1$ for $m \in \mathbb{N}$, $m \geq 3$.

Article [13] states exact values and estimates on $L_\oplus$, $L_\otimes$ for arbitrary quadratic Pisot numbers, i.e. solutions to

$$\begin{aligned} x^2 &= mx + n\,, && m, n \in \mathbb{N},\ m \geq n, \\ x^2 &= mx - n\,, && m, n \in \mathbb{N},\ m \geq n + 2\,. \end{aligned}$$

The first cubic case was studied by Messaoudi. In [16] he shows for the Tribonacci numeration system ($\beta^3 = \beta^2 + \beta + 1$) that $L_\otimes \leq 9$, and states the conjecture of Arnoux that $L_\otimes = 3$. In [15] he improves the upper bound to $L_\otimes \leq 6$. In [13] it is shown that $4 \leq L_\otimes \leq 5$ and $5 \leq L_\oplus \leq 6$. Final exact value $L_\oplus = 5$ has been given with elegant proof by Bernat [7].

Paper [5] treats a class of cubic Pisot units, solutions to $x^3 = mx^2 + x + 1$, $m \in \mathbb{N}$, generalization of the Tribonacci case. Several other cubic cases are studied in [7].

5 Conclusions and outlook

Several objectives concerning the arithmetics on non-standard numeration systems can be pursued in future. Among them the challenging problem of algebraic characterization of numbers with finiteness property, providing bounds on the length of fractional part appearing in the arithmetic operations for larger classes of bases β, or proving or disproving existence of numbers β, for which $L_\oplus$, $L_\otimes$ are infinite. One can also concentrate on designing arithmetic algorithms working with infinite eventually periodic β-expansions.

A complete solution to these problems appears to be very difficult to achieve. The methods which were fruitful on the field of non-standard numeration systems are mainly number-theoretical; some progress has been brought also by computer experiments. Recently, also combinatorial approach proved to be useful [6]. The study of properties of the infinite word u_β corresponding to the Parry base β, such as factor and palindromic complexity, or balance properties may be of help in the search for results on arithmetics in the numeration system with the base β.

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