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Stabilizace a bezpodmínečnost v Banachových prostorech

Stabilization and unconditionality in Banach spaces

Summary

In this talk we recall some classical and more recent problems related to stabilization of functions on spheres of Banach spaces, unconditionality of bases, and decomposability (*e.g.* the distortion problem, the unconditional basic sequence problem). We show how these problems interconnect and then outline the progress that has happened in this area in the 1990's.

Souhrn

V této přednášce připomeneme některé klasické i novější problémy týkající se stabilizace funkcí na jednotkových sférách Banachových prostorů, bezpodmínečnosti bází a rozložitelnosti (např. problém distorze, problém bezpodmínečné basické posloupnosti). Ukážeme, jak jsou tyto problémy navzájem propojeny, a pak načrtneme, jakého pokroku bylo v této oblasti dosaženo v 90. letech 20. století.

klíčová slova:
Banachův prostor, báze, bezpodmínečnost, stabilizace, distorze, rozložitelnost.

keywords:
Banach space, basis, unconditionality, stabilization, distortion, decomposability.

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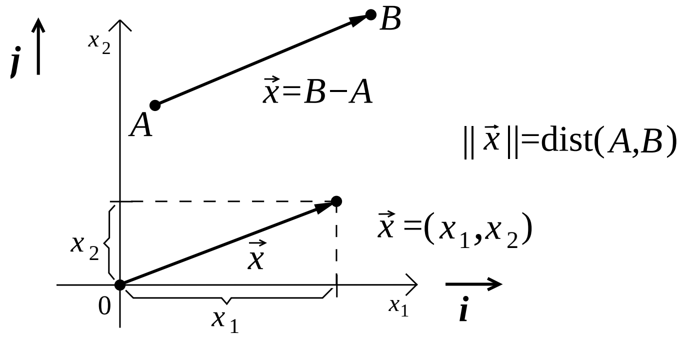
In this talk we will first introduce the notion of a Banach space, focusing on properties that we will need later. Then we will introduce some problems that were posed in the "classical times" of the Banach space theory and show some connections between them. In the last part of this talk we will show how these problems were solved in the 1990's and talk about some ramifications.

1. Introduction

Our starting point is the traditional two-dimensional Euclidean space. In this space we have a linear structure—we have vectors that can be added and multiplied by numbers—and a topological structure—namely, we have the notion of distance. We also have the natural basis **i**, **j** and vectors can be coded by their coordinates.



In particular, we can calculate the distance of two points by the Pythagorean formula. The linearity of this space (distances do not change by shifting a picture) means that we can reduce the problem of distance to the question of magnitude of vectors.

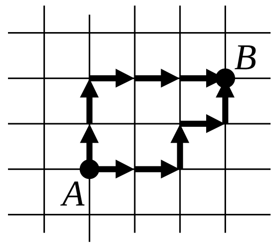
This brings us to the notion of a *Banach space*. By a Banach space we mean a linear space X endowed with a *norm*, which is a magnitude of vectors that satisfies certain requirements, for instance symmetry about the origin, the triangle inequality, and completeness (Cauchy sequences must converge). The norm of a Banach space is usually denoted $\|x\|$ for a vector $x \in X$ (note that we did not use the "vector arrow" $\vec{x}$, since it is not customary in the Banach space theory).

The above example of the 2-dimensional Euclidean space is one of the simplest examples of a Banach space. Its norm is denoted by $\|\cdot\|_2$ (the subscript "2" indicates that the norm is Euclidean) and is given by

$$\|(x_1, x_2)\|_2 = \sqrt{|x_1|^2 + |x_2|^2} = \left(|x_1|^2 + |x_2|^2\right)^{\frac{1}{2}}$$

for any vector $x = (x_1, x_2)$ from $I\!R^2$. The Banach space formed by endowing $I\!R^2$ with this Euclidean norm is denoted ℓ_2^2, the lower index indicates the Euclidean space and the upper index its dimension.

There are other ways of measuring distance in the plane. For instance, in a city with a regular grid of streets, the practical distance to some destination is given by how much we have to travel, which is the sum of the distances in the two directions given by the streets.



Thus we have another norm on $I\!R^2$:

$$\|(x_1, x_2)\|_1 = |x_1| + |x_2|.$$

The two-dimensional space endowed with this norm is denoted ℓ_1^2.

In some settings, the "natural" approach would ask which of the two displacements in the basic two directions is maximal, and take this as the distance. Thus we get the norm

$$\|(x_1, x_2)\|_\infty = \max(|x_1|, |x_2|).$$

The resulting space is called ℓ_∞^2.

These three norms are actually just special cases of a more general norm. The most popular norms on $I\!R^2$ are given by the following formula, where p is a parameter from the interval $[1, \infty)$:

$$\|(x_1, x_2)\|_p = \big(|x_1|^p + |x_2|^p\big)^{\frac{1}{p}}.$$

While the Euclidean norm and $\|\cdot\|_1$ clearly fit this pattern, the norm $\|\cdot\|_\infty$ can be considered a limit case; precisely, for $x \in I\!R^2$, $\|x\|_\infty$ is the limit of $\|x\|_p$ as $p \to \infty$. Thus we get the scale of the classical spaces ℓ_p^2 for $p \in [1, \infty]$. The scale is nicely ordered, if $q > p$, then $\|x\|_q \le \|x\|_p$ for all $x \in I\!R^2$.

This shows that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are extreme in the scale of the p-norms. One can even prove that for any norm $\|\cdot\|$ on $I\!R^2$ that is equal to 1 when applied to $(1, 0)$ and $(0, 1)$, necessarily

$$\|x\| \le \|x\|_1 \text{ for all } x \in I\!R^2.$$

We will return to comparison of norms shortly.

We just saw that given a Banach space, there can be many norms on the underlying linear space, each creating a different Banach space. The notation is a bit vague at this. When we say "a Banach space X", the letter X represents the Banach space, that is, both the underlying linear space and the norm given on it; however, sometimes we use X just for the underlying linear space, for instance when we start considering another norm on it. Usually this is not a problem, but when we want to be precise, we use the notation $(X, \|\cdot\|)$ to indicate precisely the linear space and the norm that defines the Banach space in question. For instance, we can write $\ell_2^2 = (I\!R^2, \|\cdot\|_2)$ or $\ell_p^2 = (I\!R^2, \|\cdot\|_p)$.

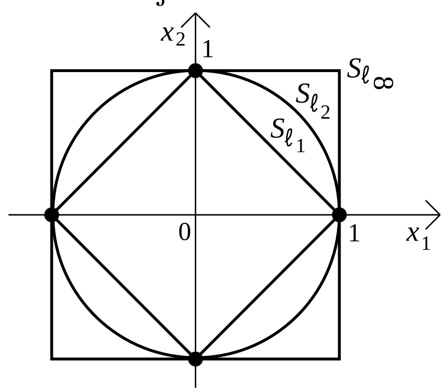
It can be proved that every norm $\|\cdot\|$ of a Banach space X is uniquely determined by the shape of its *unit sphere*

$$S_X = \{x \in X;\ \|x\| = 1\}.$$

Indeed, given any point $x \in X$, we can find the intersection of S_X and the ray going from the origin through x, that is, we can find $a > 0$ so that $ax \in S_X$. Then $\|ax\| = 1$, so from the properties of the norm it follows that $\|x\| = \frac{1}{a}$.

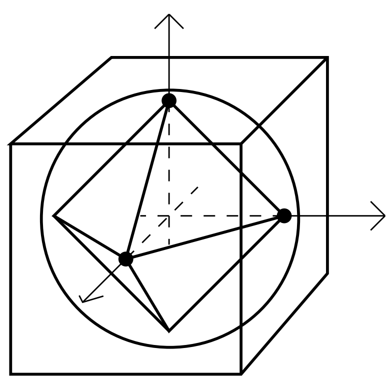
Relationships between norms can be also seen through their unit spheres. In particular, given two norms $\|\cdot\|$, $|||\cdot|||$ on a Banach space X, the norm $|||\cdot|||$ dominates the norm $\|\cdot\|$, that is, $|||x||| \geq \|x\|$ for all $x \in X$, if and only if the unit sphere of $(X, |||\cdot|||)$ is "inside" the unit sphere of $(X, \|\cdot\|)$.

Here are the unit spheres of the three norms on $\mathbb{R}^2$ that we started our talk with; note how they fit inside one another, illustrating the relationship between inequalities and placement of unit spheres that we have just discussed:



Note that spheres of the two extreme norms, $\|\cdot\|_1$ and $\|\cdot\|_\infty$, have corners and parts that are flat. These two norms are indeed somewhat special in the scale of norms $\|\cdot\|_p$ and some properties of the corresponding spaces are different.

All this can be easily generalized to higher dimensions, obtaining spaces $\ell_p^N = (\mathbb{R}^N, \|\cdot\|_p)$ for $p \in [1, \infty]$. The unit sphere of ℓ_2^3 is the usual sphere, the unit sphere of ℓ_∞^3 is a cube, the unit sphere of ℓ_∞^3 is an octahedron.



Finally, one can pass to infinite dimension. Let $p \in [1, \infty)$ be given. Consider the space of all sequences $x = (x_i) = \{x_1, x_2, \dots\}$ such that $\sum |x_i|^p$ is finite. These sequences form a linear space and we can endow it with the norm

$$\|(x_i)\|_p = \Big(\sum_{i=1}^{\infty} |x_i|^p\Big)^{\frac{1}{p}}.$$

The resulting Banach space is called ℓ_p. Similarly we generalize ℓ_∞^N to the space ℓ_∞ with a supremum norm, but this space is rather large and it is customary to work with the space c_0 instead. It consists of all sequences (x_i) such that $x_i \to 0$ and uses the norm

$$\|(x_i)\|_\infty = \max\{|x_i|;\ i = 1, 2, 3, \dots\}.$$

These classical sequence spaces have many of the properties that we discussed in two dimensions. In particular, the norms of ℓ_1 and c_0 are extreme (the unit sphere of c_0 can be thought of as an infinite-dimensional cube). For $1 < p < \infty$, the spheres have no "corners" and are therefore "nicer", and the "best" norm of all is the Euclidean norm $\|\cdot\|_2$ of the infinite-dimensional Euclidean space ℓ_2; the unit sphere of ℓ_2 is an infinite-dimensional sphere. For obvious reasons we do not present a picture.

There is another reason why the Euclidean spaces are considered best from many points of view. Both ℓ_2^N and ℓ_2 are Hilbert spaces, unlike the other sequence spaces.

Note that there is a big difference between the finite-dimensional and infinite-dimensional ℓ_p spaces. Given a natural number N, then $\|\cdot\|_p$ are different norms on the same underlying linear space $\mathbb{R}^N$; that is, if $p \neq q$, then ℓ_p^N and ℓ_q^N differ just in norm. When we pass to the infinite case, we find that every space ℓ_p is different. If $p < q$, then there are sequences $x = (x_i)$ for which $\sum |x_i|^q$ is finite, but $\sum |x_i|^p$ is not. It means that such sequences are in ℓ_q but not in ℓ_p; that is, even the underlying linear spaces are different. Thus when we write ℓ_p, it has a richer meaning than ℓ_p^N, since we also specify a special linear space.

However, for $p < q$ one has the following: The linear space ℓ_p is a subset of ℓ_q. Thus some comparison of norms is possible, as both norms are defined on ℓ_p and it makes sense to say that $\|x\|_q \le \|x\|_p$ for all $x \in \ell_p$. In this sense we can say that the inequalities between norms $\|\cdot\|_p$ that we mentioned for $\mathbb{R}^2$ are also true for spaces ℓ_p.

Remark: While in this talk it will be more useful to use the unit spheres S_X, usually one prefers to work with the *unit ball*

$$B_X = \{x \in X;\ \|x\| \le 1\}.$$

The main advantage of the unit ball is that it is a more "substantial" set than the unit sphere, which makes some things easier. For instance, having two norms $\|\cdot\|$, $|\!|\!|\cdot|\!|\!|$ on the same Banach space X, the inequality $|\!|\!|x|\!|\!| \ge \|x\|$ for all $x \in X$ is equivalent to the inclusion $B_{(X,|\!|\!|\cdot|\!|\!|)} \subseteq B_{(X,\|\cdot\|)}$, which is a precise mathematical statement, unlike the rather vague "being inside" that we used for unit spheres.

The unit ball also defines the norm uniquely, they are connected by the equality

$$\|x\| = \inf\Big\{\frac{1}{a};\ a > 0,\ ax \in B_X\Big\}.$$

Here we can say even more. Given a norm on a Banach space, then its unit ball is a closed, symmetric, and convex set. Conversely, every closed, symmetric, and convex set is a unit ball of some norm, in fact of the norm defined by the above equality.

2. Some Problems

When defining the classical spaces ℓ_p, we represented elements of these spaces as sequences of coordinates. This is related to the fact that every classical space ℓ_p (or c_0) has a Schauder basis. By a Schauder basis of a Banach space X we mean a sequence $\{e_i\}_{i=1}^{N,\infty}$ (finite or countably infinite) of vectors such that every element x of X can be expressed as a sum $x = \sum_i a_i e_i$, and the coefficients a_i—the coordinates of x with respect to this basis—are unique. Every element of such a space can then be thought of in terms of its coordinates (a_i), so a Schauder basis provides the space with a coordinate system. The spaces ℓ_p have a natural Schauder basis—the canonical unit vector basis, it was actually built into it in the definition. However, in general one cannot expect to have a natural basis, since the elements of a given Banach space may not resemble classical vectors at all. For instance, in applications we frequently use Banach spaces of functions. Yet, if we can find a Schauder basis in such spaces, we can treat them as spaces of sequences whenever it is helpful.

Note that there is a big difference between an algebraic basis and a Schauder basis. In the definition of an algebraic basis, only finite sums are allowed in the decomposition; consequently, in "large" spaces algebraic bases must be uncountable to yield all vectors. One advantage of this notion is that every linear space has an algebraic basis. However, they are not very helpful when investigating Banach spaces. On the other hand, a Schauder basis can be only at most countable, but we allow for infinite sums in the decomposition (the convergence is taken with respect to the norm of the space). So Schauder bases are typically "smaller" than algebraic bases, but unfortunately they are not guaranteed to exist, as we will shortly see.

If we can find a finite basis (algebraic or Schauder), it means that the Banach space in question is finite-dimensional and these two notions of basis agree. One can show that such a Banach space is in fact very close to the Euclidean space of an appropriate dimension. Therefore, some questions have very simple answers. This is the reason why in many problems we only consider infinite-dimensional Banach spaces. We will do the same, in this talk we will always assume that Banach spaces and also all subspaces that we mention are infinite-dimensional, unless stated otherwise. It should be noted that not only we always mean "infinite-dimensional" when talking about subspaces here, but also that unless explicitly stated, all subspaces are meant to be linear and "closed", which means that they by themselves are Banach spaces when endowed with the norm inherited from the whole original space.

The existence of a Schauder basis restricts rather severely the number of vectors that such a Banach space can have. If a Banach space has a Schauder basis, it must necessarily be *separable*, which is a notion that describes a "size" of a space; for many practical purposes, countably many vectors already represent the whole separable space. Consequently, a Banach space that is not separable—which has too many vectors—cannot have a Schauder basis.

Banach, the forefather of the Banach space theory, has asked in the early 1930's the natural question whether every separable Banach space has a Schauder basis ([B]). There was a hope for a positive answer, as all concrete spaces that were studied and/or appeared in applications had some sort of a natural basis. This question was open for a long time, until in 1972 Enflo constructed a Banach space that does not have a Schauder basis ([E]). On the other hand, it was already known in the 1930's that given a Banach space, one can always find a subspace of it that already has a Schauder basis.

While the basis problem remained open for so long, it was clear that one could not expect too much of a given Banach space. However, in many situations one could improve one's situation by passing to a subspace that is "nicer", for instance has a Schauder basis (as we already mentioned). Passing to a "better" subspace thus became a very useful tool. Since the spaces ℓ_p have very good properties, already in the 1930's the following question was asked:

(P1) *Is it true that every infinitely-dimensional Banach space has a subspace that is isomorphic to ("looks like") c_0 or ℓ_p for some $p \in [1, \infty)$?*

We chose to emphasize that we only consider infinite-dimensional spaces in our problems, because every finite-dimensional Banach space already is isomorphic to some ℓ_2^N. Thus the question would be trivial for finite dimension.

Since all known concrete examples of Banach spaces contained some ℓ_p, there was a hope that the answer to (P1) might be positive.

As the theory developed, a special kind of Schauder basis turned out to be very useful. It is called the *unconditional basis.*

Definition
A Schauder basis $\{e_i\}$ of a Banach space X is *unconditional* if there is a number C such that for all vectors $x = \sum a_i e_i \in X$ and arbitrary signs $\varepsilon_i = \pm 1$ one has

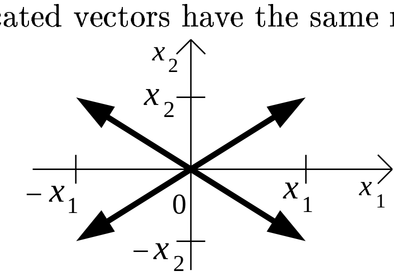
$$\left\|\sum \varepsilon_i a_i e_i\right\| \leq C \left\|\sum a_i e_i\right\| = C\|x\|.$$

The "best" (smallest) constant C that can be used above is called the *unconditional basis constant* and is a measure of "quality" of the unconditional basis.

Recall that in the Euclidean two-dimensional space ℓ_2^2 we indeed have that

$$\|(x_1, x_2)\|_2 = \|(\pm x_1, \pm x_2)\|_2,$$

which proves that its canonical basis is unconditional with $C = 1$, the best possible constant. In the picture, all indicated vectors have the same magnitude.



This shows that unconditionality is actually a kind of symmetry. The norm of a vector does not change much (we have a general control over its size with C) if we consider symmetries with respect to the coordinate axes.

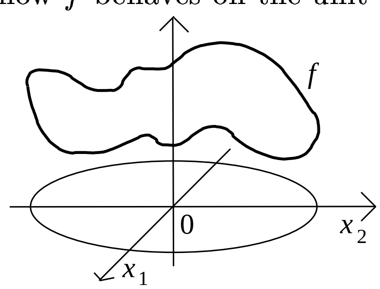
It follows right from the definition that the canonical bases of the classical spaces ℓ_p are unconditional (with $C = 1$). However, even the classical function spaces L_1 and $C[0,1]$ were shown not to have an unconditional basis, so one cannot expect to have it in general. Could we at least pass to a subspace that has such a basis?

(P2) (the "unconditional basic sequence problem") *Is it true that every infinitely-dimensional Banach space has an infinitely-dimensional subspace that has an unconditional basis?*

Note that a positive answer to (P1) would immediately imply a positive answer to (P2).

We again emphasized infinite dimensions to make the question meaningful; every basis of a finite-dimensional Banach space is automatically unconditional.

For the next two problems we have to start somewhere else. Consider again the two-dimensional Euclidean space. Let f be a uniformly continuous function on $\mathbb{R}^2$. For many purposes it is enough to know how f behaves on the unit sphere, so we have this picture:

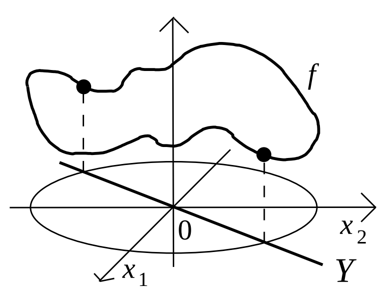


We are interested in how much the function changes:

Definition
Let X be a Banach space, let f be a uniformly continuous function defined on the sphere S_X of X. For a subspace Y of X we define the *oscillation* of f on Y by

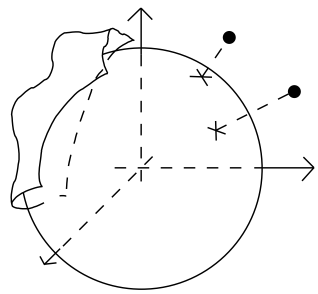
$$\operatorname*{osc}_Y(f) = \sup\{f(x) - f(y);\ x, y \in S_Y\}.$$

It is a standard fact that in the example of $\mathbb{R}^2$ above, there is a line through the origin so that when we look at f at the points where this line intersects the unit sphere, the values of f are the same.

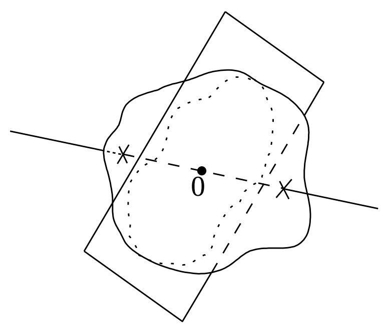


In the Banach space language, given a uniformly continuous function f on the unit sphere of ℓ_2^2, there is a subspace Y of $\mathbb{R}^2$ such that $\operatorname{osc}_Y(f) = 0$.

Now consider a function defined on the unit sphere of the three-dimensional Euclidean space. We can, for instance, imagine the function to be the elevation of the real surface of the Earth at points of the ideal spherical surface.



Experience suggests that it is not possible to find a plane going through the origin (a two-dimensional subspace of $\mathbb{R}^3$) so that the elevation function becomes constant on this subspace, that is, so that the oscillation decreases to zero. However, it is easy to find such a one-dimensional subspace (a line through the origin), because it is easy to find a direction such that on its opposite ends the elevation is the same.



In fact, for every finite-dimensional Banach space X, for every uniformly continuous function f on S_X there is a subspace Y of X such that $\operatorname{osc}_Y(f) = 0$. Now this is easy, we simply find an appropriate one-dimensional subspace—a line through the origin. However, it was proved that one can find such subspaces of large dimensions; precisely, given the dimension of the original space X, there is a number m so that for every 1-Lipschitz function f on S_X there exists a "stabilizing" subspace Y whose dimension is at least m, and m grows to infinity as the dimension of X grows to infinity! ([M2], [MS])

So for spaces of finite dimension the situation is clear again. What if we start with a space of infinite dimension and want the subspace Y on which the function is constant to be also infinite-dimensional? Then we will not succeed (apart from some trivial examples). For infinite dimensions we therefore need a different approach.

Definition
We say that a function f defined on the unit sphere S_X of an infinite-dimensional Banach space X *stabilizes* if for every infinite-dimensional subspace Y of X and every $\varepsilon > 0$ there is an infinite-dimensional subspace Z of Y such that $\operatorname{osc}_Z(f) < \varepsilon$.

Now we have the following problem:

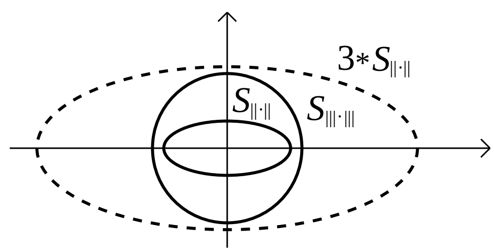
(P3) *Is it true that on every infinite-dimensional space, every uniformly continuous function stabilizes?*

In fact, it is equivalent to ask this question just about Lipschitz functions. This question also stayed open for a long time, and it did not get any easier even when it was restricted to very special uniformly continuous functions: equivalent norms.

Definition
Let $\|\cdot\|$, $|||\cdot|||$ be two norms on the same underlying linear space X. We say that they are *equivalent* if there are constants $C, D > 0$ so that for all x from X we have

$$\|x\| \le C|||x||| \text{ and } |||x||| \le D\|x\|.$$

In other words, the two norms are equivalent if the unit sphere $S_{\|\cdot\|}$ of one norm can be enlarged by a finite ratio so that the unit sphere $S_{|||\cdot|||}$ of the other norm is now inside it, and vice versa. In the following picture we see the situation corresponding to the inequalities $|||x||| \le \|x\| \le 3|||x|||$ for all $x \in X$.



In a typical situation we are given a Banach space X with a norm $\|\cdot\|$ and we start considering also another norm $|||\cdot|||$ on the same space, usually with some additional useful properties (this procedure is called *renorming*). If this new norm is equivalent, then the new space $(X, |||\cdot|||)$ preserves many of the properties of the original space, the spaces are similar. This is very important, because if we solve some problem using the new, better norm, the solution can be often also used with the original norm.

If this new norm is not equivalent, then the renormed space has nothing in common with the original one and for all practical purposes it is a different space. Therefore, when renorming a space, we always consider an equivalent norm.

It should be noted that on a given finite-dimensional space, all norms are equivalent to the Euclidean one; this is just another reason why finite-dimensional spaces are for many purposes "not interesting".

Now consider (an infinite-dimensional) Banach space $(X, \|\cdot\|)$ with a new (equivalent) norm $|||\cdot|||$. This norm is a uniformly continuous function (even Lipschitz) on the unit sphere S_X of the original norm, so we can ask whether it can be stabilized. Actually, for norms we usually ask less. For functions, to stabilize meant that we can make their oscillation small in every subspace. With equivalent norms, we would be happy if we can make the oscillation small just somewhere. That is, given $\varepsilon > 0$, we want to find a subspace (infinitely-dimensional as usual here) on which the equivalent norm oscillates by at most ε.

If this can be done, it would mean that there is a subspace of X on which the new norm is almost just a multiple of the original norm—in other words, on this subspace the original and the renormed space are essentially the same. Since passing to a better subspace is one of our favorite tricks, this situation would be very helpful.

If this is not possible, if the new equivalent norm oscillates "everywhere" on the unit sphere S_X of the original norm, then the geometry of the new space is different from the original geometry, we would say that the original space was distorted by the new norm. We will now make it formal, with a little change: For norms we prefer to measure their oscillation using a ratio rather than difference.

Definition
Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space, let $|\!|\!|\cdot|\!|\!|$ be an equivalent norm on X. We say that $|\!|\!|\cdot|\!|\!|$ is λ-*distorted* for some $\lambda > 1$ if for every infinite-dimensional subspace Y of X we have

$$\lambda \le \sup\Big\{\frac{|\!|\!|x|\!|\!|}{|\!|\!|y|\!|\!|};\ x, y \in S_{(X,\|\cdot\|)}\Big\}.$$

If there is some $\lambda > 1$ as above for the given equivalent norm $|\!|\!|\cdot|\!|\!|$, the norm would be called *distorted*, and the largest constant λ that satisfies the above property is called the *level of distortion* of the norm $|\!|\!|\cdot|\!|\!|$.

Note that for such a distorted norm, its oscillation cannot be made almost zero by passing to some subspace, that is, our attempt to stabilize it as described before this last definition would fail. In particular, note that such a distorted equivalent norm does not stabilize as a function.

On the other hand, note that the supremum in the definition is always at least 1. If we cannot find any $\lambda > 1$ as in the definition, it means than given $\varepsilon > 0$, one could find a subspace Y of X on which the ratio $|\!|\!|x|\!|\!|/|\!|\!|y|\!|\!|$ is almost 1 (up to the ε) for all $x, y \in Y$, that is, the norm $|\!|\!|\cdot|\!|\!|$ is almost constant on Y and its oscillation almost zero. This means that the original and renormed spaces are almost the same when we look just at Y. We see that this definition exactly captures the idea that we explained above.

We say that a Banach space X is *distortable* if there is $\lambda > 1$ and an equivalent norm on X that is λ-distorted. Now we can state another question:

(P4) *Is it true that there is no distortable Banach space?*

It immediately follows that a positive answer to (P3) would also mean a positive answer to (P4). The problem (P4) is weaker than the problem (P3), but they actually coincide for the spaces ℓ_p. Precisely, *for* $1 < p < \infty$, *if* ℓ_p *is not distortable, then on this space also all uniformly continuous functions stabilize* ([OS]).

Both problems stayed unsolved for a long time, although some interesting results and insights came in the 1960's. For instance, James proved that *the spaces* c_0 *and* ℓ_1 *are not distortable* ([J]). However, recall that these two spaces are "extreme" in the scale of ℓ_p spaces and this was heavily used in the proof; thus these techniques could not be used for ℓ_p with $1 < p < \infty$ or other spaces.

In 1971, Milman proved that *if on a given Banach space all uniformly continuous functions stabilize, then it must contain as a subspace a copy of* c_0 *or* ℓ_p *for some* $1 < p < \infty$

([M1]). Thus a positive answer to (P3) would imply a positive answer to (P1). In fact, he proved somewhat more. *If a Banach space X has no distortable subspace, then it must contain as a subspace a copy of c_0 or ℓ_p for some $1 < p < \infty$.* Since it was thought that copies of ℓ_p can be found, this result supported the hope that (P3) and (P4) might have a positive answer.

The first breakthrough came in 1974, when Tsirelson constructed a *space that does not contain any copy of c_0 or ℓ_p* ([T]). Thus the question (P1) was surprisingly answered in the negative, and by Milman's result also the question (P3). Furthermore, by his result, the Tsirelson space contains a subspace that is distortable, so also the question (P4) was answered in the negative. We remark that the original Tsirelson construction was modified to improve other "qualities" of this space, and eventually there came examples of spaces that were rather "nice" and still did not contain any copy of the classical sequence spaces.

In the light of these results, the original questions now changed:

(P3') *What Banach spaces have the property that all uniformly continuous functions on them stabilize?*

(P4') (the "distortion problem") *Are the spaces ℓ_p, for $1 < p < \infty$, distortable?*

In 1989 Odell directly proved that the whole *Tsirelson space is distortable* (see *e.g.* [O]). Moreover, he showed that one can modify its construction to obtain spaces T_k with the following property: Given natural number k, there is an equivalent norm on T_k with level of distortion at least k. Thus one can find arbitrarily large distortions, but for every level of distortion, one had to look at a specific space. Could there be one universal space so that for every $\lambda > 1$ it would have an equivalent norm that is λ-distortable?

(P4") *Is there an arbitrarily distortable space?*

This was the state of art at the beginning of the 1990's.

3. The 1990's Developments

In 1991, Schlumprecht has constructed a *Banach space that is arbitrarily distortable* ([S]), thus answering (P4") in the positive. He used some ideas of Tsirelson and modified them in a crucial way. His space also had other interesting properties that are beyond the scope of this talk.

His construction was used a year later to answer (P4') also in the positive. With Odell they proved that *all spaces ℓ_p for $1 < p < \infty$ are arbitrarily distortable* ([OS]). They also showed that *ℓ_1 admits a uniformly continuous function on its unit sphere that does not stabilize.* Recall that it was proved earlier that all equivalent norms on ℓ_1 do stabilize, but unlike the case of $1 < p < \infty$, here the problems of equivalent norm stabilization and uniformly continuous function stabilization are not equivalent. Recall that also for c_0 the two problems are not equivalent and c_0 was proved to be not distortable, but here the situation was somewhat different: In 1991 in an unrelated work, Gowers showed that *on c_0, all uniformly continuous functions stabilize* ([G1]). This settled the stabilization situation for spaces ℓ_p. We remark that in 1995, Hájek with the speaker proved that all polynomials, and hence all sufficiently smooth functions, stabilize on ℓ_p for $1 \leq p < \infty$ ([HH]).

The results by Odell and Schlumprecht also helped to answer the question (P3'). From the work of Milman ([M1]) it follows that if all uniformly continuous functions on a Banach space X stabilize, then the space has a subspace isomorphic to c_0 or some ℓ_p. But since the spaces ℓ_p for $1 < p < \infty$ are distortable and ℓ_1 admits a uniformly continuous function that does not stabilize, it consequently follows that such a space must contain copies of c_0 in all its subspaces.

We also obtain the following dichotomy: *Every Banach space X contains a copy of c_0 or ℓ_1, or a subspace that is distortable.*

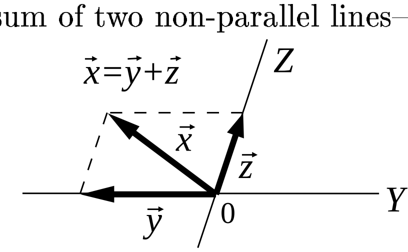
The new methods also lead to solution of problem (P2). It started when Gowers and Maurey observed that the distortions in the Schlumprecht space are special, and that the existence of a "large special distortion" means that the space can be renormed so that even if there was a subspace with an unconditional basis, its unconditionality constant must be bad. This shows another connection between the topics we cover here. They used this observation to construct a space (in 1992) that has no subspace with an unconditional basis ([GM]), thus answering (P2) in the negative.

So while in 1990 one might still hope that general Banach spaces cannot be all that bad, the new examples showed that one cannot really hope in general for very much. And since even the classical spaces ℓ_p were found to be distortable, it would seem that distortability is normal rather than an exception.

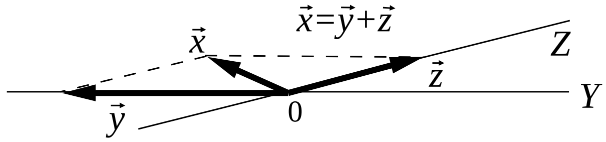
The example of Gowers and Maurey was later adopted by various authors to answer further questions. Among others, the speaker constructed a Banach space that does not have any subspace with the GL property, which is a weaker notion than having an unconditional basis ([H]).

The Gowers-Maurey space has also many other properties, one of them was that of *hereditary indecomposability.* First we introduce a few notions:

A general vector space is said to be decomposed into a sum of two linear subspaces if $X = Y + Z$, where the sum is taken pointwise; that is, every vector from X can be decomposed as a sum of a vector from Y and a vector from Z. The sum $X = Y + Z$ is called *direct* if for all $x \in X$ such decompositions are unique; equivalently, the intersection of the subspaces Y and Z is the trivial space $\{0\}$. For instance, the two-dimensional plane can be expressed as a direct sum of two non-parallel lines—subspaces of dimension one.



In this two-dimensional picture, direct sum always means that there is a positive angle between the two lines; this applies to all finite dimensions. This positive angle is very important for the decomposition. The closer this angle is to the right angle, the better the decomposition from the point of view of topology. Conversely, if the angle is close to zero, the decomposition does not behave well, because even a small vector may have very large components.



If X is an infinite-dimensional Banach space decomposed into a direct sum $Y+Z$, where Y, Z are infinite-dimensional linear subspaces (not necessarily closed), it may happen that although the two subspaces intersect only at the origin, the angle is zero in a limit sense. This has very unpleasant consequences, since as the above picture suggests, components of small vectors have norms that grow beyond any bound. If it does not happen, we say that the decomposition is *topological*, which is usually characterized by Y and Z being closed; that is, by themselves they also constitute Banach spaces.

Note that if we are given a decomposition and any of the subspaces involved is finite-dimensional, then the decomposition is necessarily topological.

Definition
Let X be an infinite-dimensional Banach space. We say that X is *indecomposable* if it cannot be expressed as a topological sum of two infinite-dimensional subspaces.

We say that X is *hereditarily indecomposable*, HI for short, if all its (closed) infinite-dimensional subspaces (including X itself) are indecomposable.

Indecomposability means that the space is a bit strange; to have an HI space is even stranger. The Gowers-Maurey space turned out to be HI ([GM]), in fact it was the first known example of such a space. This property became a key to further properties of this space; in particular, an HI space cannot have a subspace with an unconditional basis ([GM]). Later, Tomczak-Jaegermann proved that every HI space is necessarily arbitrary distortable ([TJ]), thus showing another connection between unconditionality and distortions.

Further research lead to one of the deepest structural results in this area, the Gowers' dichotomy. He proved the following ([G2]):

(G) *Every Banach space has a subspace with an unconditional basis or a subspace that is hereditarily indecomposable.*

This result shows that while the 1990's brought a host of extremely weird spaces, it also brought a deeper understanding of the structure of Banach spaces. We already mentioned one dichotomy earlier, and the important Gowers' dichotomy helped to solve another very old problem.

Recall that two Banach spaces are called *isomorphic* if there exists a bijection between them such that the bijection and its inverse are bounded linear operators. From the Banach space theory point of view, isomorphic spaces are very similar and share many of the important properties. It was known early on that the space ℓ_2 is isomorphic to all its (closed infinite-dimensional) subspaces. In the 1930's, Banach asked ([B]) whether it is the only space satisfying this property, we call it a *homogeneous space*:

(P5) (the "homogeneous space problem") *If an infinite-dimensional Banach space X is isomorphic to all its closed infinite-dimensional subspaces, is it necessarily isomorphic to ℓ_2?*

This question also stayed open for a long time. In the end the *positive answer* to this problem was obtained by combining three independent results, each of which used different methods. One of them is the Gowers' dichotomy, another is due to Gowers and Maurey ([GM]):

(GM) *If a Banach space is HI, then it is not isomorphic to any proper subspace of itself.*

The third theorem, due to Komorowski and Tomczak-Jaegermann ([KTJ]), states that every Banach space of cotype $q < \infty$ contains either a copy of ℓ_2 or a subspace without an unconditional basis. Since an earlier result by Szarek showed that homogeneous spaces must be of some cotype $q < \infty$, we in particular have the following statement:

(KTJ) *If X is a homogeneous Banach space, then it is isomorphic to ℓ_2 or it does not have an unconditional basis.*

Now we are ready to solve the homogeneous space problem.

Let X be a homogeneous (infinite-dimensional) Banach space. By (G) it either has a subspace that is HI, or a subspace that has an unconditional basis.

If it had an HI subspace, then by homogeneity, the whole space would have to be HI, so by (GM) it could not be isomorphic to its proper subspaces, which contradicts its homogeneity.

So necessarily, X contains a subspace with an unconditional basis. By homogeneity, the whole space X has an unconditional basis, so by (KTJ), X must be isomorphic to ℓ_2 and the claim is proved.

4. Some Open Problems

Despite the great advances of the 1990's, some problems remain open. Concerning distortions, all distorted spaces that were constructed in the 1990's have also arbitrary distortions. Is it possible to have a distortable Banach space with bounded distortions; that is, a Banach space that is distorted, but there is a bound D so that all equivalent norms on this space have level of distortion at most D? In particular, this question remains open for the original Tsirelson space, which is known to reach distortion levels up to 2. Precisely, for every $\varepsilon > 0$ there is an equivalent norm with level of distortion at least $2 - \varepsilon$. However, all efforts to construct distorted norms of higher level of distortion failed, so the Tsirelson space is a natural candidate for a space of bounded distortion.

This position is further strengthened by another result. The research of the 1990's also lead to investigation of asymptotic structures. In particular, asymptotic ℓ_p spaces are spaces whose finite-dimensional subspaces are close to subspaces of ℓ_p, assuming that we only look very far from the beginning of the basis. Roughly speaking, these spaces are similar to ℓ_p in the limit sense, at ends of their bases. Milman and Tomczak-Jaegermann proved that a Banach space with bounded distortions must necessarily be an asymptotic ℓ_p-space [MTJ], and the Tsirelson space happens to be one.

The precise relationship between hereditary indecomposability, distortions and unconditionality is also unclear at the moment. Although there has not been another breakthrough in the last few years, the research continues.

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- "Krivine p's in quotients of Banach spaces", AMS Meeting in Louisville, Kentucky, USA, March 1998.
- "Representing ℓ_p on quotients", AMS Meeting in Lawrenceville, New Jersey, USA, October 1996.
- "On the construction of the space without the Gordon–Lewis property hereditarily", the Young Researchers Seminar, MSRI, Berkeley, California, USA, April 1996.
- "A Banach space whose subspaces fail the Gordon–Lewis property", Concentration in Infinite-Dimensional Convex Geometry, MSRI, Berkeley, California, USA, February 1996.
- "Recent developments in distortions and unconditionality", Operator Theory on the Prairies Symposium, University of Alberta, Edmonton, Canada, October 1995.