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SUITABLE WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

Summary

This lecture deals with the results on the local regularity of the suitable weak solutions of the Navier-Stokes equations.

In the first section we present the definition of a weak solution and its basic properties. We stress that the regularity of a weak solution is closely connected with the existence of the singular points of this solution. Therefore, in the second section we define the notion of a suitable weak solution. It is known that suitable weak solutions have only "few" interior singular points which opens the way to the effective study of the local regularity of these solutions. The third section is devoted to the survey of some results of this type. In the forth section we prove in more detail two criterions on local regularity of suitable weak solutions based on the local regularity of two components of the vorticity vector and local regularity of the pressure. In the last section we present two results on the boundary regularity of suitable weak solutions.

Souhrn

Přednáška se zabývá prezentací výsledků o lokální regularitě vhodných slabých řešení Navierových - Stokesových rovnic.

V první části definujeme pojem slabého řešení a diskutujeme jeho základní vlastnosti. Regularita slabého řešení je těsně spojena s existencí singulárních bodů tohoto řešení. Ve druhé části proto definujeme pojem vhodného slabého řešení. Je známo, že vhodná slabá řešení mají pouze "málo" singulárních bodů, což otevírá cestu k efektivnějšímu studiu lokální regularity těchto řešení. Třetí část je věnována přehledu některých výsledků tohoto typu. Ve čtvrté části dokazujeme detailněji dvě kritéria o lokální regularitě vhodných slabých řešení založená na lokální regularitě dvou složek vektoru vířivosti a na lokální regularitě tlaku. V poslední části přednášky pak uvádíme dva výsledky týkající se hraniční regularity vhodných slabých řešení.

Klíčová slova

Navierovy-Stokesovy rovnice, slabé řešení, vhodné slabé řešení, lokální regularita řešení, energetická nerovnost, zobecněná energetická nerovnost

Key words

Navier-Stokes equations, weak solution, suitable weak solution, local regularity of solution, energy inequality, generalized energy inequality

Contents

1. Definition and some basic properties of weak solutions

Let $\Omega = \mathbb{R}^3$ or Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, let $T > 0$ and $Q_T = \Omega \times (0, T)$. Consider the Navier-Stokes initial-boundary value problem describing the evolution of the velocity $u(x, t)$ and the pressure $p(x, t)$ in Q_T:

$$\frac{\partial u}{\partial t} - \nu\Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } Q_T, \tag{1}$$

$$\text{div } u = 0 \quad \text{in } Q_T, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T) \text{ if } \Omega \text{ is a bounded domain}, \tag{3}$$

$$u|_{t=0} = u_0, \tag{4}$$

where $\nu > 0$ is the viscosity coefficient. The initial data u_0 satisfy the compatibility conditions $u_0|_{\partial\Omega} = 0$ and div $u_0 = 0$. Let us stress that the condition (3) does not apply if $\Omega = \mathbb{R}^3$.

In what follows, we use the standard notation for the Lebesgue and Sobolev spaces (L^p and $W^{k,p}$, respectively) and for the corresponding norms ($\| \cdot \|_p$ and $\| \cdot \|_{k,p}$, respectively). To simplify the notation we often write $\int f$ instead of $\int_\Omega f(x)\ dx$ or $\int f(x)\ dx$ instead of $\int f(x, t)\ dx$. We do not distinguish between $(L^p)^m$ and L^p. The outer normal vector is denoted by n. In the paper c stands for a generic constant.

As is usual in the standard theory of the Navier-Stokes equations let us define $D(\Omega) = \{\psi \in C_0^\infty(\Omega); \nabla \cdot \psi = 0 \text{ in } \Omega\}$, let $L^2_\sigma(\Omega)$ be the completion of $D(\Omega)$ in $L^2(\Omega)$ and let $W^{1,2}_{0,\sigma}(\Omega)$ be the completion of $D(\Omega)$ in $W^{1,2}(\Omega)$. Let us define also $D_T = \{\varphi \in C_0^\infty(\Omega \times [0, T)); \nabla \cdot \varphi = 0 \text{ in } \Omega \times [0, T)\}$.

Definition 0.1. *Let $u_0 \in L^2_\sigma(\Omega)$. A measurable function $u : Q_T \to \mathbb{R}^3$ is called a weak solution of the problem (1)–(4) if $u \in L^2(0, T, W^{1,2}(\Omega)) \cap L^\infty(0, T, L^2_\sigma(\Omega))$ and*

$$\int_0^T \int_\Omega \left[u \cdot \frac{\partial \varphi}{\partial t} - \nu \nabla u \cdot \nabla \varphi - u \cdot \nabla u \cdot \varphi \right] dxdt = - \int_\Omega u_0 \cdot \varphi(\cdot, 0)\ dx$$

for all $\varphi \in D_T$.

Equivalent definitions can be found in [**27**] or [**6**].

The existence of weak solutions is generally known but their uniqueness and regularity remain an open problem even in the case of smooth u_0 (for all these facts see e.g. [**27**]). Having established a weak solution u, there exists a distribution p $-$ the associated pressure, so that the equation (1) holds in the sense of distributions in Q_T.

It was proved in ([**7**], Theorem 3.1) that every weak solution u and its associated pressure p have in fact these regularity properties:

$$\left\| \frac{\partial u}{\partial t} \right\|_{s,q} + \|\nabla^2 u\|_{s,q} + \|\nabla p\|_{s,q} < \infty, \tag{5}$$

for every $1 < s, q < \infty$ and $\frac{2}{s} + \frac{3}{q} = 4$, if the initial condition u_0 is sufficiently smooth (see [**7**], p.77). Moreover, the pressure p can be chosen in such a way that

$$\|p\|_{s,r} < \infty, \tag{6}$$

where $s \in (1, 2)$, $r \in (3/2, 3)$ and $\frac{2}{s} + \frac{3}{r} = 3$ and

$$\|\nabla u\|_{h,\rho} < \infty, \tag{7}$$

where $\frac{2}{h} + \frac{3}{\rho} = 3$ and $h, \rho > 1$.

Let us note that every weak solution u of (1)–(4) is a weakly continuous function from $[0, T)$ into $L^2_\sigma(\Omega)$ (see [**6**], Lemma 2.2).

A weak solution u is usually called a Leray-Hopf weak solution if the energy inequality is satisfied, that is

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds \le \|u_0\|_2^2$$

for every $t \in [0,T]$. It follows from the energy inequality that for every Leray-Hopf weak solution u

$$\lim_{t\to 0_+} \|u(t) - u_0\|_2 = 0.$$

As an example of a Leray-Hopf weak solution can serve the weak solution constructed in [**27**] (Theorem 3.1) by the Faedo-Galerkin method.

A stronger condition than the energy inequality is the so called strong energy inequality (see [**6**], (4.1)). We say that a weak solution u of (1)–(4) satisfies the strong energy inequality, if

$$\|u(t_2)\|_2^2 + 2\int_{t_1}^{t_2} \|\nabla u(s)\|_2^2 ds \le \|u(t_1)\|_2^2 \tag{8}$$

for every $t_2 \in (0,T]$ and almost every $0 \le t_1 \le t_2$.

Let us present the following result (see [**6**], Theorem 6.1 or [**27**], Theorem 3.11).

Theorem 0.2. *Let $u_0 \in W^{1,2}_{0,\sigma}(\Omega)$. Then there exists $T^* \in (0,T]$ and at least one weak solution u of (1) - (4) such that*

$$u \in L^\infty(0,T^*; W^{1,2}_{0,\sigma}(\Omega)) \cap L^2(0,T^*; W^{2,2}(\Omega)), \tag{9}$$

$$T^* \ge \frac{C\nu^3}{\|\nabla u_0\|_2^4} \tag{10}$$

and C depends only on Ω.

The weak solution u from Theorem 0.2 satisfies the energy inequality and is unique in the class of Leray-Hopf weak solutions on $(0,T^*)$ (see for example [**6**], Theorem 4.2). If we were able to show that $T^* = T$ then it would meant that we have a unique regular solution of (1)–(4) with the initial condition $u_0 \in W^{1,2}_{0,\sigma}(\Omega)$ on the whole interval $(0,T)$. Hovever, it is not known, whether the solution u from Theorem 0.2 is really from the space $L^\infty(0,T; W^{1,2}_{0,\sigma}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega))$. In fact the following possibility is not excluded: there exists $t_0 \in (0,T)$ such that

$$\lim_{t\to t_0^-} \|u(t)\|_{1,2} = \infty. \tag{11}$$

In such a case we say that the solution u has a blow-up at the time t_0. Let us present now the notion of regular and singular points. A point $(x_0,t_0) \in \overline{\Omega} \times (0,T)$ is called a regular point of u if u is essentially bounded in a space-time neighbourhood U of (x_0,t_0), that is if $u \in L^\infty(U)$. A point $(x_0,t_0) \in \overline{\Omega} \times (0,T)$ is called singular if it is not regular.

If u is a weak solution of (1)-(4) and $u \in L^\infty(a,b; W^{1,2}_{0,\sigma}(\Omega)) \cap L^2(a,b; W^{2,2}(\Omega))$ for $(a,b) \subseteq (0,T)$ then it follows from [**22**] and [**5**] that there exists no singular point of u in $\overline{\Omega}\times(a,b)$. Vice versa, if (11) holds then it is possible to show that u has at least one singular point in $\overline{\Omega}\times\{t_0\}$. Indeed, let us suppose that u does not have any singular point in $\overline{\Omega}\times\{t_0\}$. It follows then from the definition of regular points of u that $u \in L^\infty(\Omega \times (t_0-\delta, t_0+\delta))$ for some $\delta > 0$ sufficiently small. By the use of Theorem 8.7 from [**24**] one gets that $u \in L^\infty(t_0-\delta/2, t_0+\delta/2; W^{1,2}_{0,\sigma}(\Omega))$ which is contradictory to (11).

2. Suitable weak solutions and the set of singular points

We see that there exists a close connection between the blow-up of a weak solution u and its singular points. In what follows we will define a suitable weak solution of (1)-(4) and present results concerning the set of all singular points of this solution.

Definition 0.3. *The pair (u, p) is called a suitable weak solutions of (1)–(4) if u is a weak solution of (1)–(4) from Definition 0.1 and the so called generalized energy inequality is satisfied, that is*

$$2\nu \int_0^T \int_\Omega |\nabla u|^2 \phi \le \int_0^T \int_\Omega \Big[|u|^2 \Big(\frac{\partial \phi}{\partial t} + \nu \Delta \phi \Big) + (|u|^2 + 2p) u \cdot \nabla \phi \Big] \tag{12}$$

for every non-negative real-valued function $\phi \in C_0^\infty(Q_T)$.

Let us remark that it is not necessary to present the usual assumption that $p \in L^{5/4}(Q_T)$ (see [**1**], p. 780) since it is satisfied for every weak solution, as follows from (6).

There is also an equivalent form of (12):

$$\begin{aligned} &\int_{\Omega \times \{t\}} |u|^2 \phi + 2\nu \int_0^t \int_\Omega |\nabla u|^2 \phi \le \\ &\int_0^t \int_\Omega |u|^2 \Big(\frac{\partial \phi}{\partial t} + \nu \Delta \phi \Big) + \int_0^t \int_\Omega (|u|^2 + 2p) u \cdot \nabla \phi, \end{aligned} \tag{13}$$

which holds for every non-negative real-valued function $\phi \in C_0^\infty(Q_T)$ and every $t \in [0, T]$.

The notion of the suitable weak solution was firstly used by Scheffer (see [**20**]). The suitable weak solutions were thoroughly studied in [**1**], where their existence was proved under the assumption that $u_0 \in L^2_\sigma(\Omega)$. In the case of a bounded Ω it is further required that $u_0 \in W^{2/5}_{5/4}(\Omega)$ (see [**1**], Theorem A').

It was proved in [**10**] that if the initial condition u_0 is sufficiently smooth (e.g. $u_0 \in W^{1,2}_{0,\sigma}(\Omega)$) then there exists a suitable weak solution u of (1)–(4) which satisfies the generalized energy inequality for every smooth test function. More precisely, the following theorem can be proved:

Theorem 0.4. *Let $u_0 \in W^{1,2}_{0,\sigma}(\Omega)$. Then there exists a suitable weak solution (u, p) of (1)–(4). Furthermore, the function $u : [0, T] \to L^2_\sigma(\Omega)$ is weakly continuous,*

$$p \in L^{s,r}, \text{ where } \frac{2}{s} + \frac{3}{r} = 3, \ s \in (1, 2), \ r \in (3/2, 3)$$

and

$$\begin{aligned} &\int_{\Omega \times \{t_2\}} |u|^2 \phi + 2\nu \int_{t_1}^{t_2} \int_\Omega |\nabla u|^2 \phi \le \int_{\Omega \times \{t_1\}} |u|^2 \phi \\ &+ \int_{t_1}^{t_2} \int_\Omega |u|^2 \Big(\frac{\partial \phi}{\partial t} + \nu \Delta \phi \Big) + \int_{t_1}^{t_2} \int_\Omega (|u|^2 u + 2pu) \nabla \phi \end{aligned} \tag{14}$$

holds for every $\phi \in C^\infty(\overline{Q}_T), \phi \ge 0$, for every $t_2 \in [0, T]$ and almost every $t_1 \in [0, t_2]$. Moreover, (14) holds also for $t_1 = 0$ and every $t_2 \in [0, T]$.

By the choice of $\phi \equiv 1$ in (14) we see that u satisfies the strong energy inequality which does not seem to be known for the suitable weak solutions satisfying the generalized energy inequality (13).

We are now interested in what can be said about the set $S(u)$ of all singular points from $\Omega \times (0, T)$ of a suitable weak solution u. It was proved in [**1**] that one-dimensional Hausdorff measure of $S(u)$ is equal to zero or more strongly, one-dimensional parabolic measure of $S(u)$ is equal to zero. For the definition of both measures see ([**1**], p.783). This result is a straightforward consequence of the following regularity criterion proved in [**1**] (Proposition 2).

Theorem 0.5. *Let (u,p) be a suitable weak solution of (1) - (4). Then there exists a positive number ε_* with the following property. Assume that for a point $z_0 = (x_0, t_0) \in Q_T$ the inequality*

$$\limsup_{r\to 0_+} \frac{1}{r} \int_{Q^*(z_0,r)} |\nabla u|^2 < \varepsilon_* \tag{15}$$

holds, where $Q^*(z_0, r) = B(x_0, r) \times (t_0 - \frac{7}{8}r^2, t_0 + \frac{1}{8}r^2)$. Then z_0 is a regular point of u.

The generalized energy inequality and a suitable decomposition of pressure play key roles in the proof of Theorem 0.5. To derive then that Theorem 0.5 implies that one-dimensional parabolic measure of $S(u)$ is equal to zero, is relatively easy: it suffices to use the covering Lemma 6.1 from [**1**] and the fact that $u \in L^2(0, T, W^{1,2}(\Omega))$. Let us note here that there exists a close relation between the global regularity of a weak solution u and the size of the set of all singular points of u.

Theorem 0.5 was improved in [**4**]:

Definition 0.6. *Let h be an increasing continuous function on $(0, 1]$, $\lim_{t\to 0_+} h(t) = 0$ and $h(1) = 1$. For fixed $\delta > 0$ and $E \subseteq \mathbb{R}^3 \times R$ let $L(\delta)$ be the family of all coverings $\{Q_{r_i}(x_i, t_i)\}$ of E with $0 < r_i \le \delta$, that is $E \subseteq \cup Q_{r_i}(x_i, t_i)$, where $Q_r(x, t) = B_r(x) \times (t - r^2, t + r^2)$. Put*

$$\Psi_\delta(E, h) = \inf_{L(\delta)} (\sum_i h(r_i))$$

and set

$$\Lambda(E, h) = \lim_{\delta\to 0_+} \Psi_\delta(E, h).$$

$\Lambda(E, h)$ is called the parabolic measure with respect to h.

If $h(t) = t^\alpha$, $\alpha > 0$, then $\Lambda_\alpha(E) = \Lambda(E, h)$ is the standard $\alpha-$ dimensional parabolic measure.

Theorem 0.7. *Let (u,p) be a suitable weak solution of (1) - (4). Then $\Lambda(S(u), h) = 0$, if $h(t) = t\left(\ln \frac{e}{t}\right)^\sigma$ and $0 \le \sigma < \frac{3}{44}$.*

The criterion presented in [**13**] is the generalization of Theorem 0.5 in two aspects. Firstly, it uses $Q(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0)$ instead of $Q^*(z_0, r)$ and secondly, z_0 is not only a regular point of u but u is a Hölder continuous function in a space-time neighborhood of z_0. Let us note that the same result was also proved in [**21**] for the case of $x_0 \in \partial\Omega$ and $\partial\Omega \cap B_r(x_0)$ lying in a plane for some $r > 0$. In [**21**] the result was obtained under the assumption that the generalized energy inequality holds up to the boundary.

3. Survey of some results on local regularity of suitable weak solutions

We see from the results presented that the set $S(u)$ of all interior singular points of a suitable weak solution is small in a certain sense. This fact is useful and can serve as a basis for the proofs of many results on the local regularity of suitable weak solutions. Before presenting these results let us describe a certain general frame.

The set $S(u)$ is a closed set relatively to Q_T. It has one useful consequence. Let $K \subseteq \Omega$ be a compact set. It is possible to write

$$(0, T) = \cup_{\gamma\in\Gamma}(\alpha_\gamma, \beta_\gamma) \cup G, \tag{16}$$

where Γ is at most countable, $(\alpha_\gamma, \beta_\gamma)$ are disjoint intervals in $(0, T)$ and the Lebesgue measure of G is equal to zero. Further, G consists of all times in which the solution u has a singular point in K. It means that there are no singular points of u in $K \times \cup_{\gamma\in\Gamma}(\alpha_\gamma, \beta_\gamma)$.

Let us suppose now that D is a sub-domain of Q_T and $(x_0, t_0) \in D$ is an arbitrary point. If (u, p) is a suitable weak solution of (1)–(4) and if one wants to show that (x_0, t_0) is a regular point of u then it is possible to suppose (for a detailed discussion see [**16**], [**11**], [**12**], [**17**], [**18**] and also [**21**] and [**9**]) that the following conditions are fulfilled:

There exist positive numbers $\varepsilon_1 < \varepsilon_2$ and τ such that $\overline{B_2} \times [t_0 - \tau, t_0 + \tau] \subseteq D$ and $(\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau] \cap S(u) = \emptyset$, where $B_i = B_{\varepsilon_i}(x_0)$, $i = 1, 2$ and $S(u)$ is a set of all singular points of u from Q_T. Further, there are no singular points of u in $\overline{B_2} \times [t_0 - \tau, t_0)$, u and all its space derivatives are continuous in $(\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau]$ and $\frac{\partial u}{\partial t}$ and p and all their space derivatives are in $L^\beta((\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau])$ for every $\beta \in (1, 2)$. Moreover, if $\Omega = \mathbb{R}^3$ then $\frac{\partial u}{\partial t}$ and p and all their space derivatives are in $L^\infty((\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau])$.

On the basis of the just presented results it is possible to prove many partial regularity results for the suitable weak solutions. The basic form of these results is asfollows: we suppose some additional regularity assumption on u or on the derivatives of u or possibly on the pressure p to be fulfilled in D. Using then this assumption together with the results just presented, it is then often possible to show the non-existence of singular points of u in D. Let us present briefly a few results of this type which were proved recently.

The following result was proved in [**18**].

Theorem 0.8. *Let (u, p) (where $u = (u_1, u_2, u_3)$) be a suitable weak solution of (1) - (4). Suppose that there exists a sub-domain D of Q_T such that u_3 is essentially bounded in D. Then u has no singular points in D.*

The result was generalized in [**16**]:

Theorem 0.9. *Let (u, p) (where $u = (u_1, u_2, u_3)$) be a suitable weak solution of (1) - (4). Suppose that there exists a sub-domain D of Q_T such that $u_3 \in L^{r,s}(D)$, where $r \in [4, \infty], s \in (6, \infty], \frac{2}{r} + \frac{3}{s} \le \frac{1}{2}$. Then u has no singular points in D.*

The result from [**11**]:

Theorem 0.10. *Let (u, p) (where $u = (u_1, u_2, u_3)$) be a suitable weak solution of (1) - (4). Suppose that D is a sub-domain of Ω and $u_1, u_2 \in L^2(0, T; W^{1,3}(D))$. Then u has no singular points in D.*

The result from [**25**]:

Theorem 0.11. *Let (u, p) (where $u = (u_1, u_2, u_3)$) be a suitable weak solution of (1) - (4). Suppose that D is a sub-domain of Ω and $u_3 \in L^4(0, T; W^{1,3}(D))$. Then u has no singular points in D.*

The following result from [**17**] has an interesting geometric interpretation.:

Theorem 0.12. *Let (u, p) be a suitable weak solution of (1) - (4), D is a sub-domain of Q_T and $\xi_1 \le \xi_2 \le \xi_3$ are the eigenvalues of the tensor $\frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. Suppose further that $\xi_2 = \xi_2^I + \xi_2^{II}$, where*

$$(i) \quad \xi_2^I \in L^{\infty,a}_{loc}(D) \text{ for some } a > \frac{3}{2} \text{ and}$$
$$(ii) \quad (\xi_2^{II})_+ \in L^{1,\infty}_{loc}(D),$$

$(\xi_2^{II})_+$ denoting the positive part of ξ_2^{II}. Then u has no singular points in D.

Further results concerning the local regularity of the suitable weak solutions were described and proved in [**15**] and [**23**]. Let us present here one simple example:

Let $(x_0, t_0) \in Q_T$, $\rho > 0$, $r > 0$ and $\sigma_0 = r^2/\rho^2$. Let us denote

$$U_r^\rho \;=\; \{(\mathbf{x}, t) \in Q_T; t_0 - \sigma_0 < t < t_0, \rho\sqrt{t_0 - t} < |\mathbf{x} - \mathbf{x}_0| < r\}.$$

Theorem 0.13. *Let us suppose that (u, p) is a suitable weak solution to (1) - (4), $(\mathbf{x}_0, t_0) \in Q_T$, $\rho \in (0, \sqrt{2\nu})$ and*

$$|u(x,t)| \le c, \quad |p(x,t)| \le c \quad \text{in } U_r^\rho$$

for some c and $r > 0$. Then (x_0, t_0) is a regular point of u.

4. Proofs of some results on local regularity of suitable weak solutions

The main goal of this section is to show that some criterions on regularity of Leray-Hopf weak solutions of the Cauchy problem for the Navier-Stokes equations hold also locally. We will present three examples of such local regularity results. The results are formulated for the suitable weak solutions. The boundary integrals appearing during the computations are then easily controllable due to the boundedness of the velocity u and all its space derivatives near the boundary. Firstly, we begin with the regularity criterion proved recently in [**19**] for the Cauchy problem and show that this criterion holds locally as well. It is also the case for some other results and we present two more examples concerning the regularity of weak solutions stemming from the regularity of two components of the vorticity [**2**] or from the regularity of the pressure [**3**]. Finally, we present a regularity criterion near the boundary. Its proof is based on the results in [**14**].

We will suppose in this section that $D \subseteq Q_T$ is an open set and $(x_0, t_0) \in D$ is an arbitrary point. If (u, p) is a suitable weak solution of (1)–(4) and if one wants to show that (x_0, t_0) is a regular point of u then it is possible to suppose (for a detailed discussion see [**16**] or also [**11**], [**12**], [**17**], [**18**]) that the following conditions are fulfilled:

There exist positive numbers $\varepsilon_1 < \varepsilon_2$ and τ such that $\overline{B_2} \times [t_0 - \tau, t_0 + \tau] \subseteq D$ and $(\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau] \cap S(u) = \emptyset$, where $B_i = B_{\varepsilon_i}(x_0)$, $i = 1, 2$ and $S(u)$ is a set of all singular points of u from Q_T. Further, there are no singular points of u in $\overline{B_2} \times [t_0 - \tau, t_0)$, u and all its space derivatives are continuous in $(\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau]$ and $\frac{\partial u}{\partial t}$ and p and all their space derivatives are in $L^\beta((\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau])$ for every $\beta \in (1, 2)$. Moreover, if $\Omega = \mathbb{R}^3$ then $\frac{\partial u}{\partial t}$ and p and all their space derivatives are in $L^\infty((\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau])$.

For further purposes we denote $B_3 = B_{\varepsilon_3}(x_0)$, where $\varepsilon_3 = (\varepsilon_1 + \varepsilon_2)/2$.

In the chapter we will use the following regularity criterion proved in [**13**] (see Theorem 2.2). Let us present here only a simplified version for $f \equiv 0$.

Theorem 0.14. *Let (u, p) be a suitable weak solution of (1) - (4). Then there exists a positive number ε_* with the following property. Assume that for a point $z_0 = (x_0, t_0) \in Q_T$ the inequality*

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q(z_0, r)} |\nabla u|^2 < \varepsilon_* \tag{17}$$

holds, where $Q(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0)$. Then z_0 is a regular point of u.

In fact, Theorem 2.2 in [**13**] is still stronger than Theorem 0.14. It even says that the velocity u is Hölder continuous function in some space-time neighborhood of z_0.

The famous criterion proved in [**1**] (Proposition 2), is weaker then Theorem 0.14, since it uses $Q^*(z_0, r) = B(x_0, r) \times (t_0 - \frac{7}{8}r^2, t_0 + \frac{1}{8}r^2)$ instead of $Q(z_0, r)$.

Let us note that an analogical result to Theorem 0.14 was proved in [**21**] for the case of $x_0 \in \partial\Omega$ and $\partial\Omega \cap B_r(x_0)$ lying in a plane for some $r > 0$.

As an inspiration for this section served the following regularity criterion proved recently by M. Pokorný in [**19**].

Theorem 0.15. *Let $u_0 \in W^{1,2}(\mathbb{R}^3)$ with div $u_0 = 0$, $\Omega = \mathbb{R}^3$ and let u be a weak solution of the Navier-Stokes equations (1) - (4) satisfying the energy inequality. Assume moreover that $\nabla u_3 \in L^\alpha(0,T;L^\gamma)$ with $\frac{2}{\alpha}+\frac{3}{\gamma} \le \frac{3}{2}$, $\frac{4}{3} \le \alpha \le \infty$, $2 \le \gamma \le \infty$. Then u and the corresponding pressure p is the smooth solution of the Navier-Stokes equations, i.e. $u \in L^\infty(0,T;W^{1,2}(\mathbb{R}^3)) \cap L^2(0,T;W^{2,2}(\mathbb{R}^3))$, $\nabla p \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$. Moreover, $u \in C^\infty([\delta,T)\times\mathbb{R}^3)$ and $p \in C^\infty([\delta,T)\times\mathbb{R}^3)$ for any small positive δ.*

It is possible to prove the following local version of Theorem 0.15.

Theorem 0.16. *Let (u,p) be a suitable weak solution of (1) - (4) and $D \subseteq Q_T$ be an open set. Assume moreover that $\nabla u_3 \in L^{\alpha,\gamma}(D)$ with $\frac{2}{\alpha}+\frac{3}{\gamma} \le \frac{3}{2}$, $\frac{4}{3} \le \alpha \le \infty$, $2 \le \gamma \le \infty$. Then u has no singular points in D.*

We are not going to prove Theorem 0.16 here. Its proof uses the same ideas as the proof of the following Theorem 0.17.

Let $\omega = \text{curl } u = (\omega_1,\omega_2,\omega_3) = (\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}, \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1})$ denote the vorticity field. The two-component vorticity field is denoted $\tilde\omega = \omega_1 e_1 + \omega_2 e_2$, where $e_1 = (1,0,0), e_2 = (0,1,0)$. The following theorem is a local version of Theorem 1 proved in [**2**].

Theorem 0.17. *Let (u,p) be a suitable weak solution of (1) - (4). Let $D \subseteq Q_T$ be an open set and $\tilde\omega \in L^{\alpha,\gamma}(D)$ with $\frac{2}{\alpha}+\frac{3}{\gamma} \le 2$, $1 < \alpha < \infty$, $\frac{3}{2} < \gamma < \infty$ or the norm of $\tilde\omega$ in the space $L^{\infty,\frac{3}{2}}(D)$ is sufficiently small. Then u has no singular points in D.*

Proof: The proof of Theorem 0.17 follows the lines of the proof in [**2**]. The boundary integrals can be handled because the suitable weak solution is considered. We begin with the equation

$$\frac{\partial\omega}{\partial t} - \nu\Delta\omega + u\cdot\nabla\omega - \omega\cdot\nabla u = 0. \tag{18}$$

Multiplying (18) by ω and integrating it over B_3 we get that

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_2^2 + \nu\|\nabla\omega(t)\|_2^2 = \int_{B_3}\omega\cdot\nabla u\cdot\omega + \int_{\partial B_3}\frac{\partial\omega}{\partial n}\cdot\omega - \frac{1}{2}\int_{\partial B_3} u\cdot n|\omega|^2 \tag{19}$$

holds for almost every $t \in (t_0-\tau, t_0)$. The last two boundary integral can be estimated by a constant c independent of time.

We estimate the integral $\int_{B_3}\omega\cdot\nabla u\cdot\omega$. We can express u by means of ω:

$$\begin{aligned} u(x) &= \frac{1}{4\pi}\int_{B_2}\frac{\text{rot }\omega(\xi)}{|x-\xi|}\,d\xi \\ &+\frac{1}{4\pi}\int_{\partial B_2}\frac{\partial u}{\partial n_\xi}(\xi)\frac{1}{|x-\xi|}\,d_\xi S - \frac{1}{4\pi}\int_{\partial B_2}u(\xi)\frac{\partial}{\partial\xi}\frac{1}{|x-\xi|}\,d\xi. \end{aligned} \tag{20}$$

Equation (20) holds for every $t \in (t_0-\tau,t_0)$ and every $x \in B_2$. Therefore, $u = \bar u + \bar{\bar u}$, where $\bar u$ is defined by the first integral on the right hand side of (20) and $\bar{\bar u}$ is defined by sum of the second and the third integral on the right hand side of (20). Since $D_x^\gamma \bar{\bar u} \in L^\infty(B_3\times(t_0-\tau,t_0))$ for every multiindex $\gamma = (\gamma_1,\gamma_2,\gamma_3)$, $\gamma_i \ge 0$, $i = 1,2,3$, it is possible to notice that there are no problems with $\bar{\bar u}$ in the following procedures. Therefore, we work only with $\bar u$ and for the sake of simplicity denote it as u.

From the definition of u we have for every $x \in B_3$ that

$$\begin{aligned} \frac{\partial u_i}{\partial x_j}(x) &= \lim_{\varepsilon\to 0+}\int_{\partial B_\varepsilon(x)} \frac{\partial}{\partial \xi_j}\frac{1}{|x-\xi|}\epsilon_{ilk} n_k(\xi)\omega_l(\xi) d_\xi S \\ &- \int_{\partial B_2} \frac{\partial}{\partial \xi_j}\frac{1}{|x-\xi|}\epsilon_{ilk} n_k(\xi)\omega_l(\xi) d_\xi S \\ &+ \lim_{\varepsilon\to 0+}\int_{B_{2,\varepsilon}(x)} \frac{\partial^2}{\partial \xi_j \partial \xi_k}\frac{1}{|x-\xi|}\epsilon_{ilk}\omega_l(\xi) d\xi \\ &= I_1(x) + I_2(x) + I_3(x), \end{aligned} \tag{21}$$

where $B_{2,\varepsilon}(x) = B_2 \setminus B_\varepsilon(x)$ and ϵ_{ijk} is the Levi-Civita's tensor. After short computation one can get

$$I_1(x) = -\frac{4\pi}{3}\epsilon_{ijl}\omega_l(x)$$

and if we put $I_1(x)$ into the integral $\int_{B_3} \omega\cdot\nabla u\cdot\omega$ instead of ∇u we get

$$\int_{B_3} \omega\cdot\nabla u\cdot\omega = -\frac{4\pi}{3}\epsilon_{ijl}\int_{B_3}\omega_i\omega_j\omega_l = 0. \tag{22}$$

Since $|x-\xi| > (\varepsilon_2-\varepsilon_1)/2$ for every $x \in B_3$ and every $\xi \in \partial B_2$, I_2 is bounded in $B_3 \times (t_0-\tau, t_0)$ and for the sake of simplicity we do not consider this term any further.

If we put $I_3(x)$ into the integral $\int_{B_3}\omega\cdot\nabla u\cdot\omega$ instead of ∇u and decompose $\omega = \tilde{\omega}+\tilde{\tilde{\omega}}$, $\tilde{\tilde{\omega}} = (0,0,\omega_3)$, we get

$$\begin{aligned} &\int_{B_3}\omega_j(x)\cdot\Big(\lim_{\varepsilon\to 0+}\int_{B_{2,\varepsilon}(x)}\frac{\partial^2}{\partial\xi_j\partial\xi_k}\frac{1}{|x-\xi|}\epsilon_{ilk}\omega_l(\xi)d\xi\Big)\cdot\omega_i(x)\,dx \\ &= \int_{B_3}\omega_j P_{ij}(\omega)\omega_i = \int_{B_3}\omega_j P_{ij}(\tilde{\omega})\tilde{\omega}_i + \int_{B_3}\omega_j P_{ij}(\tilde{\omega})\tilde{\tilde{\omega}}_i \\ &+ \int_{B_3}\omega_j P_{ij}(\tilde{\tilde{\omega}})\tilde{\omega}_i + \int_{B_3}\omega_j P_{ij}(\tilde{\tilde{\omega}})\tilde{\tilde{\omega}}_i. \end{aligned} \tag{23}$$

$P(\cdot) = (P_{ij}(\cdot))_{i,j=1}^3$ denotes the singular integral operator defined by the third integral in (21). The last integral is equal to zero. The remaining three integrals on the right hand side of (23) can be estimated by

$$\begin{aligned} &\int_{B_3}|\omega||P(\tilde{\omega})||\tilde{\tilde{\omega}}| + \int_{B_3}|\omega||P(\tilde{\omega})||\tilde{\omega}| + \int_{B_3}|\omega||P(\tilde{\tilde{\omega}})||\tilde{\omega}| \\ &\le c\int_{B_3}|\omega|^2|P(\tilde{\omega})| + c\int_{B_3}|\omega||P(\tilde{\tilde{\omega}})||\tilde{\omega}| = J_1 + J_2. \end{aligned}$$

We have by the Hölder inequality, the Calderon-Zygmund inequality, the interpolation inequality, the Sobolev inequality and the Young inequality that

$$J_1 \le \|P(\tilde{\omega})\|_\gamma \|\omega\|^2_{\frac{2\gamma}{\gamma-1}} \le c\|\tilde{\omega}\|_\gamma \|\omega\|_2^{\frac{2\gamma-3}{\gamma}}\|\nabla\omega\|_2^{\frac{3}{\gamma}} \le \frac{1}{4}\nu\|\nabla\omega\|_2^2 + C\|\tilde{\omega}\|_\gamma^\alpha\|\omega\|_2^2. \tag{24}$$

The same estimate can be obtained for J_2:

$$J_2 \le \frac{1}{4}\nu\|\nabla\omega\|_2^2 + C\|\tilde{\omega}\|_\gamma^\alpha\|\omega\|_2^2. \tag{25}$$

If $\alpha = \infty$ and $\gamma = \frac{3}{2}$ then both J_1 and J_2 can be estimated by

$$C\|\tilde{\omega}\|_{\frac{3}{2}}\|\nabla\omega\|_2^2. \tag{26}$$

We get from (19) - (26) that

$$\frac{d}{dt}\|\omega(t)\|_2^2 + \nu\|\nabla\omega(t)\|_2^2 \le C\|\tilde{\omega}\|_\gamma^\alpha\|\omega\|_2^2 + C \tag{27}$$

and by the Gronwall lemma we have

$$\omega \in L^\infty(t_0-\tau, t_0; L^2(B_3)) \cap L^2(t_0-\tau, t_0; W^{1,2}(B_3)). \tag{28}$$

We can write for every $x \in B_3$ that

$$\begin{aligned} u(x) &= \frac{1}{4\pi}\int_{B_2} \nabla_\xi \frac{1}{|x-\xi|} \times \omega(\xi)\, d\xi \\ &+ \frac{1}{4\pi}\int_{\partial B_2} \frac{1}{|x-\xi|}\Big(\omega(\xi)\times n(\xi) + \frac{\partial u}{\partial n}(\xi)\Big)\, d_\xi S - \frac{1}{4\pi}\int_{\partial B_2} \frac{\partial}{\partial n_\xi}\frac{1}{|x-\xi|} u(\xi)\, d_\xi S. \end{aligned} \tag{29}$$

The boundary integrals on the right hand side of (29) cause no problems because they are from $L^\infty(t_0-\tau, t_0; L^\infty(B_3))$. Applying now the famous results (see e.g. [**8**]) on the first integral in (29) we get that

$$u \in L^\infty(t_0-\tau, t_0; L^6(B_3))$$

and from this we are going to derive that

$$u \in L^\infty(t_0-\tau, t_0; W^{1,2}(B_3)). \tag{30}$$

In fact we will show a stronger result which we will need further in the proof of Theorem 0.18: if $u \in L^\infty(t_0-\tau, t_0; L^s(B_3))$, $s \in (3,6]$, then $u \in L^\infty(t_0-\tau, t_0; W^{1,2}(B_3))$. Let us multiply the equation (1) by $-\Delta u$, integrate it over B_3 and use the integration by parts. We get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \nu\|\Delta u\|_2^2 = \int_{B_3} u\cdot\nabla u\cdot\Delta u + \int_{\partial B_3} p\, n\cdot\Delta u + \int_{\partial B_3} n\cdot\nabla u\cdot\frac{\partial u}{\partial t}. \tag{31}$$

The last two boundary integrals are from $L^\beta(t_0-\tau, t_0)$ for any $\beta \in (1,2)$. Let us estimate the integral $\int_{B_3} u\cdot\nabla u\cdot\Delta u$.

$$\begin{aligned} |\int_{B_3} u\cdot\nabla u\cdot\Delta u| &\le \|u\|_s\|\nabla u\|_{\frac{2s}{s-2}}\|\Delta u\|_2 \le \|u\|_s\|\nabla u\|_2^{\frac{s-3}{s}}\|\nabla u\|_6^{\frac{3}{s}}\|\Delta u\|_2 \\ &\le c\|u\|_s\|\nabla u\|_2^{\frac{s-3}{s}}\|\nabla^2 u\|_2^{\frac{s+3}{s}} \le \frac{\nu}{2}\|\nabla^2 u\|_2^2 + c\|u\|_s^{\frac{2s}{s-3}}\|\nabla u\|_2^2, \end{aligned} \tag{32}$$

where we used the Hölder inequality, the interpolation inequality, the Sobolev inequality and the Young inequality. (30) now follows from (31) and (32). It means that the assumption (17) is fulfilled and Theorem 0.14 gives that (x_0, t_0) is a regular point of u. Since (x_0, t_0) was an arbitrary point in D, Theorem 0.17 is proved. •

We will now turn our attention to the paper [**3**]. The main result of this paper is the following theorem:

Theorem 0.18. *Let $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $q > 3$ with div $u_0 = 0$ in the sense of distributions and $\Omega = \mathbb{R}^3$. Suppose that u is a Leray-Hopf weak solution of (1) - (4) in $[0,T)$. If $p \in L^{\alpha,\gamma}(\mathbb{R}^3)$ with $\frac{2}{\alpha}+\frac{3}{\gamma} < 2$ and $1 < \alpha \le \infty$, $\frac{3}{2} < \gamma < \infty$, or $p \in L^{1,\infty}$, or else $\|p\|_{L^{\infty,3/2}}$ is sufficiently small, then u is a regular solution in $[0,T)$.*

For some α, γ it is possible to prove a local version of Theorem 0.18 - see Theorem 0.20. In the proof of Theorem 0.20 we will use the following Gronwall lemma (see [**26**], p. 88).

Lemma 0.19. *Let g, h and y be locally integrable nonnegative functions on $[0,\infty)$ that satisfy the differential inequality*

$$y'(t) \le g(t)y(t) + h(t) \text{ on } [0,\infty),\ y(0) = y_0.$$

Let the function $y'(t)$ be also locally integrable. Then

$$y(t) \le y(0) \exp\Big(\int_0^t g(\tau)\ d\tau\Big) + \int_0^t h(s) \exp\Big(- \int_t^s g(\tau)\ d\tau\Big)\ ds,\ t \ge 0.$$

Theorem 0.20. *Let (u, p) be a suitable weak solution of (1) - (4). Let $D \subseteq Q_T$ is an open set and $p \in L^{\alpha,\gamma}(D)$ with $1 \le \alpha \le \infty$, $\frac{3}{2} \le \gamma \le \infty$. Then u has no singular points in D if one of the following conditions is fulfilled:*

$$
\begin{aligned}
&(i) \quad \frac{2}{\alpha} + \frac{3}{\gamma} < 2,\ \gamma \in (3, \infty)\ \textit{and}\ \alpha \ge 2,\\
&(ii) \quad \frac{2}{\alpha} + \frac{3}{\gamma} < 2,\ \gamma \in (3, \infty)\ \textit{and}\ \Omega = \mathbb{R}^3,\\
&(iii) \quad \frac{2}{\alpha} + \frac{3}{\gamma} < 2\ \textit{and}\ \gamma \in (\frac{3}{2}, 3],\\
&(iv) \quad \gamma = \infty,\ \alpha = 1\ \textit{and}\ \Omega = \mathbb{R}^3,\\
&(v) \quad \gamma = \frac{3}{2},\ \alpha = \infty\ \textit{and the norm}\ \|p\|_{\infty,\frac{3}{2}}\ \textit{is sufficiently small}\ .
\end{aligned}
$$

Proof: Let $s > 3$. The proof is based on the inequality

$$
\begin{aligned}
&\frac{d}{dt} \int_{B_3} |u|^s + 2 \int_{B_3} |\nabla |u|^{s/2}|^2 \\
&\le 2(s-2) \int_{B_3} |p||u|^{\frac{s-2}{2}} |\nabla |u|^{s/2}| + \Big| \int_{\partial B_3} spu \cdot n |u|^{s-2} \Big| \\
&+ \text{non-problematic boundary integrals,}
\end{aligned}
\tag{33}
$$

which can be obtained by multiplying the equation (1) by $su|u|^{s-2}$ and using the integration by parts. The non-problematic boundary integrals and the boundary integral with p on the right hand side of (33) are from the space $L^1(t_0 - \tau, t_0)$. Using the Gronwall lemma they play the role of the function h.

We can write

$$
\begin{aligned}
p(x) &= \lim_{\varepsilon \to 0_+} \frac{1}{4\pi} \int_{B_{2,\varepsilon}(x)} u_i(y) u_j(y) \frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{|x-y|}\ dy - \frac{1}{3} |u(x)|^2 \\
&+ \frac{1}{4\pi} \int_{\partial B_2} \Big\{ \frac{\partial}{\partial y_j} (u_i(y) u_j(y)) n_i(y) \frac{1}{|x-y|} - u_i(y) u_j(y) n_j(y) \frac{\partial}{\partial y_i} \frac{1}{|x-y|} \\
&+ \frac{\partial p}{\partial n}(y) \frac{1}{|x-y|} - p(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \Big\}\ d_y S = p_1(x) + p_2(x),
\end{aligned}
\tag{34}
$$

for every $x \in B_3$, where p_2 is defined by the boundary integral in (34). Let us note that in the following estimates p_1 will be handled using the Calderon-Zygmund inequality and $p_2 \in L^\beta(t_0 - \tau, t_0; L^\infty(B_3))$ for every $\beta \in (1, 2)$ if Ω is a bounded domain and $p_2 \in L^\infty(t_0 - \tau, t_0; L^\infty(B_3))$ if $\Omega = \mathbb{R}^3$.

Let us estimate the integral $I = \int_{B_3} |p||u|^{\frac{s-2}{2}} |\nabla |u|^{s/2}|$ on the right hand side of (33). We will discus the assumptions (i) - (v).

(i) $\frac{2}{\alpha} + \frac{3}{\gamma} < 2,\ \gamma \in (3, \infty)$ and $\alpha \ge 2$.

Then there exists $s \in (3, \gamma)$, such that

$$2 - \frac{2}{\alpha} - \frac{3}{\gamma} \ge \frac{s-3}{\gamma}. \tag{35}$$

By the Hölder inequality and the Young inequality we have

$$I \leq \|p\|_s \|u\|_s^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_s^2 \|u\|_s^{s(1-\frac{2}{s})}$$
$$\leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^\alpha \|u\|_s^{s(1-\frac{2}{s})}. \tag{36}$$

Without lack of generality we can suppose that $\|u\|_s > 1$ for almost every $t \in (t_0 - \tau, t_0)$. The inequality (36) then gives that $I \leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^\alpha \|u\|_s^s$ and by the use of (33) and Lemma 0.19 we get that

$$u \in L^\infty(t_0 - \tau, t_0; L^s(B_3)). \tag{37}$$

We will show that (37) holds also in the case of conditions (ii) - (v).

(ii) $\frac{2}{\alpha} + \frac{3}{\gamma} < 2,\ \gamma \in (3, \infty)$ and $\Omega = \mathbb{R}^3$.

Again, as in (i), there exists $s \in (3, \gamma)$ such that (35) is satisfied. Then

$$I \leq \|p\|_s \|u\|_s^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \leq \|p\|_{\frac{s}{2}}^{\frac{\gamma-s}{2\gamma-s}} \|p\|_\gamma^{\frac{\gamma}{2\gamma-s}} \|u\|_s^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \leq c(I_1 + I_2), \tag{38}$$

where we used the interpolation inequality and

$$I_k = \|p_k\|_{\frac{s}{2}}^{\frac{\gamma-s}{2\gamma-s}} \|p\|_\gamma^{\frac{\gamma}{2\gamma-s}} \|u\|_s^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2,\ k = 1, 2.$$

Due to the definition of p_1 in (34) and the Calderon-Zygmund inequality we have

$$I_1 \leq \|u\|_s^{\frac{2\gamma-2s}{2\gamma-s} + \frac{s-2}{2}} \|p\|_\gamma^{\frac{\gamma}{2\gamma-s}} \|\nabla |u|^{\frac{s}{2}}\|_2 \leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s(1-\frac{2}{2\gamma-s})} \|p\|_\gamma^{\frac{2\gamma}{2\gamma-s}}$$
$$\leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s(1-\frac{2}{2\gamma-s})} \|p\|_\gamma^\alpha, \tag{39}$$

since $\frac{2\gamma}{2\gamma-s} \leq \alpha$ as a consequence of the inequality (35). Further,

$$I_2 \leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s-2} \|p\|_\gamma^{\frac{2\gamma}{2\gamma-s}} \|p_2\|_{\frac{s}{2}}^{\frac{2\gamma-2s}{2\gamma-s}}. \tag{40}$$

Since $\Omega = \mathbb{R}^3$, we know that $p_2 \in L^\infty(t_0 - \tau, t_0; L^\infty(B_3))$ and

$$I_2 \leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s-2} \|p\|_\gamma^{\frac{2\gamma}{2\gamma-s}} \leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s(1-\frac{2}{s})} \|p\|_\gamma^\alpha. \tag{41}$$

In the same way as in (i) we can conclude from (33), (38), (39), (41) and Lemma 0.19 that (37) is satisfied.

(iii) $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$ and $\gamma \in (\frac{3}{2}, 3]$.

Let us suppose firstly that

$$\gamma \in (9/4, 3]. \tag{42}$$

Then there exists $s > 3$ such that

$$\gamma > \frac{3s}{s+1} \text{ and } 2 - \frac{2}{\alpha} - \frac{3}{\gamma} \geq 1 - \frac{3}{s}. \tag{43}$$

The Hölder inequality, the interpolation inequality, the Sobolev inequality and the Young inequality give

$$I \leq \|p\|_\gamma \|u\|_{\frac{\gamma(s-2)}{\gamma-2}}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \leq \|p\|_\gamma \|u\|_s^{\frac{\gamma s-3s+\gamma}{2\gamma}} \|u\|_{3s}^{\frac{3s-3\gamma}{2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2$$
$$\leq \|p\|_\gamma \|u\|_s^{\frac{\gamma s-3s+\gamma}{2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{3s-3\gamma}{\gamma s}+1} \leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^{\frac{2\gamma s}{\gamma s-3s+3\gamma}} \|u\|_s^{s(1-\frac{2\gamma}{\gamma s-3s+3\gamma})}$$
$$\leq \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^\alpha \|u\|_s^{s(1-\frac{2\gamma}{\gamma s-3s+3\gamma})} \tag{44}$$

The last inequality follows from the inequlity $\frac{2\gamma s}{\gamma s-3s+3\gamma} \leq \alpha$ (which is a consequence of the second inequality in (43)) and (37) is satisfied.

Secondly, let

$$\gamma \in (3/2, 9/4]. \tag{45}$$

Then there exists $s > 3$ such that

$$2 - \frac{2}{\alpha} - \frac{3}{\gamma} > \frac{s-3}{\alpha}. \tag{46}$$

The Hölder inequality and the interpolation inequality give

$$I \le \|p\|_{\frac{3s}{s+1}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_{\gamma}^{\frac{\gamma(s-1)}{3s-2\gamma}} \|p\|_{\frac{3s}{2}}^{\frac{3s-\gamma-\gamma s}{3s-2\gamma}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le c(I_1 + I_2), \tag{47}$$

where

$$I_k = \|p\|_{\gamma}^{\frac{\gamma(s-1)}{3s-2\gamma}} \|p_k\|_{\frac{3s}{2}}^{\frac{3s-\gamma-\gamma s}{3s-2\gamma}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2, \ k = 1, 2.$$

By the Hölder inequality, the Sobolev inequality, the Young inequality and the fact that $\frac{\gamma s - \gamma}{2\gamma - 3} \le \alpha$ (which follows from the inequality (46)) we have

$$\begin{aligned} I_1 &\le \|p\|_{\gamma}^{\frac{\gamma(s-1)}{3s-2\gamma}} \|u\|_{3s}^{\frac{6s-2\gamma-2\gamma s}{3s-2\gamma} + \frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_{\gamma}^{\frac{\gamma(s-1)}{3s-2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{6s-8\gamma+6}{3s-2\gamma}} \\ &\le c\|p\|_{\gamma}^{\frac{\gamma(s-1)}{2\gamma-3}} + \|\nabla |u|^{\frac{s}{2}}\|_2^2 \le c\|p\|_{\gamma}^{\alpha} + \|\nabla |u|^{\frac{s}{2}}\|_2^2. \end{aligned} \tag{48}$$

Further, by the Sobolev inequality and the Young inequality

$$I_2 \le \|p\|_{\gamma}^{\frac{\gamma(s-1)}{3s-2\gamma}} \|p_2\|_{\frac{3s}{2}}^{\frac{3s-\gamma-\gamma s}{3s-2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{s-2}{s}+1} \le \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_{\gamma}^{\frac{\gamma s(s-1)}{3s-2\gamma}} \|p_2\|_{\frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}}. \tag{49}$$

Now we show the integrability in time of the second part of the right hand side of the last inequality. The Hölder inequality gives for some $\beta \in (1, 2)$ sufficiently close to 2 that

$$\int_{t_0-\tau}^{t_0} \|p\|_{\gamma}^{\frac{\gamma s(s-1)}{3s-2\gamma}} \|p_2\|_{\frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}} \, dt \le \|p\|_{\frac{\beta s\gamma(s-1)}{\beta(3s-2\gamma)-s(3s-\gamma-s\gamma)}, \gamma}^{\frac{\gamma s(s-1)}{3s-2\gamma}} \|p_2\|_{\beta, \frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}}. \tag{50}$$

Since $p_2 \in L^{\beta}(t_0 - \tau, t_0; L^{\infty}(B_3))$, we have $\|p_2\|_{\beta, \frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}} < \infty$. To show the boundedness of the integral on the left hand side of the inequality (50), it is now sufficient to realize that

$$\frac{\beta s\gamma(s-1)}{\beta(3s-2\gamma) - s(3s-\gamma-s\gamma)} \le \alpha. \tag{51}$$

We know that

$$\frac{2\gamma}{2\gamma - 3} < \alpha. \tag{52}$$

If we denote

$$g_{\gamma}(\beta, s) = \frac{\beta s\gamma(s-1)}{\beta(3s-2\gamma) - s(3s-\gamma-s\gamma)}, \tag{53}$$

then for every $\gamma \in (\frac{3}{2}, \frac{9}{4}]$, g_{γ} is a continuous function in β, s defined in an open neighbourhood of the point $(2, 3)$. It can be easily verified that

$$g_{\gamma}(2, 3) \le \frac{2\gamma}{2\gamma - 3} \quad \text{for every } \gamma \in \Big(\frac{3}{2}, \frac{9}{4}\Big]. \tag{54}$$

For β sufficiently close to 2 and s sufficiently close to 3 the inequality (51) now follows from (52), (53), (54) and the continuity of g_{γ} at the point $(2, 3)$. Therefore, the value of the integral on the left hand side of the inequality (50) is less than ∞ and (37) now follows from (33), (47), (48), (49) and Lemma 0.19.

(iv) $\gamma = \infty, \ \alpha = 1$ and $\Omega = \mathbb{R}^3$.

Then by the Hölder inequality

$$I \le \|p\|_{\infty}^{\frac{1}{2}} \|p\|_{\frac{s}{2}}^{\frac{1}{2}} \|u\|_s^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le c(I_1 + I_2), \tag{55}$$

where

$$I_k = \|p\|_{\infty}^{\frac{1}{2}} \|p_k\|_{\frac{s}{2}}^{\frac{1}{2}} \|u\|_s^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2, \ k = 1, 2.$$

In the standard way we can estimate

$$I_1 \le \|p\|_{\infty}^{\frac{1}{2}} \|u\|_s^{\frac{s}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le c\|p\|_{\infty} \|u\|_s^s + \|\nabla |u|^{\frac{s}{2}}\|_2^2, \tag{56}$$

$$I_2 \le \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_{\infty} \|p_2\|_{\frac{s}{2}} \|u\|_s^{s(1-\frac{2}{s})}. \tag{57}$$

The term $\|p\|_{\infty}\|p_2\|_{\frac{s}{2}}$ from (57) is integrable in time since $p_2 \in L^{\infty}(t_0 - \tau, t_0; L^{\infty}(B_3))$ and (37) now follows from (55), (56), (57), (33) and Lemma 0.19.

(v) $\gamma = \frac{3}{2},\ \alpha = \infty$ and the norm $\|p\|_{\infty,\frac{3}{2}}$ is sufficiently small .

Then for $s > 3$

$$I \le \|p\|_{\frac{3s}{s+1}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_{\frac{3}{2}}^{\frac{1}{2}} \|p\|_{\frac{3s}{2}}^{\frac{1}{2}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le c(I_1 + I_2), \tag{58}$$

where

$$I_k = \|p\|_{\frac{3}{2}}^{\frac{1}{2}} \|p_k\|_{\frac{3s}{2}}^{\frac{1}{2}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2, \ k = 1, 2.$$

The Calderon-Zygmund inequality gives

$$I_1 \le \|p\|_{\frac{3}{2}}^{\frac{1}{2}} \|u\|_{3s}^{\frac{s}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_{\frac{3}{2}}^{\frac{1}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2^2 \tag{59}$$

and we see that to finish the proof in this case the sufficient smallness of the norm $\|p\|_{\infty,\frac{3}{2}}$ is necessary. Further,

$$\begin{aligned} I_2 &= \|p\|_{\frac{3}{2}}^{\frac{1}{2}} \|p_2\|_{\frac{3s}{2}}^{\frac{1}{2}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_{\frac{3}{2}}^{\frac{1}{2}} \|p_2\|_{\frac{3s}{2}}^{\frac{1}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{2(s-1)}{s}} \\ &\le \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_{\frac{3}{2}}^{\frac{s}{2}} \|p_2\|_{\frac{3s}{2}}^{\frac{s}{2}}. \end{aligned} \tag{60}$$

The term $\|p\|_{\frac{3}{2}}^{\frac{s}{2}} \|p_2\|_{\frac{3s}{2}}^{\frac{s}{2}}$ is clearly integrable in time if $s \in (3, 4)$. Let us notice that for the estimate of I_2 we did not need the assumption on the smallness of the norm $\|p\|_{\infty,\frac{3}{2}}$. Again, (37) now follows from (33), (58), (59), (60) and Lemma 0.19.

Thus, (37) holds for every condition (i) - (v) for some $s > 3$. As was shown in the proof of Theorem 0.17, we then have $u \in L^{\infty}(t_0 - \tau, t_0; W^{1,2}(B_3))$ and the proof of Theorem 0.20 can be concluded by the use of Theorem 0.14. •

5. Two results on boundary regularity of suitable weak solutions

In this section $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. We present two results concerning the boundary regularity od suitable weak solutions of (1)–(4). The first result was proved recently in [**21**]:

Theorem 0.21. *Let (u, p) be a weak suitable solution of (1)–(4) from Theorem 0.4. There exists a positive number ε with the following property. Let $(x_0, t_0) \in \partial\Omega \times (0, T)$ and $B_\delta(x_0) \cap \partial\Omega$ is a part of a plane for some $\delta > 0$. Let the inequality*

$$\limsup_{r \to 0_+} \frac{1}{r} \int_{Q^+(x_0,t_0,r)} |\nabla u|^2 < \varepsilon \tag{61}$$

hold, where $Q^+(x_0,t_0,r) = [B_r(x_0)\times(t_0-r^2,t_0)]\cap Q_T$. Then (x_0,t_0) is a regular point of u. Moreover, there exists a neighbourhood U of the point (x_0,t_0) such that the function u is Hölder continuous in $U\cap\overline{Q}_T$.

The second result comes from [**14**]. For $r>0$ we denote

$$U_r = U_r(\partial\Omega) = \{x\in\Omega;\ \mathrm{dist}(x,\partial\Omega)<r\}.$$

Theorem 0.22. *Let u be a weak solution of (1) - (4) that satisfies the strong energy inequality. Let $0\le t_1<t_2\le T$ and one of the two following conditions be fulfilled:*
(i) $u\in L^p(t_1,t_2;L^{q^*}(U_r)^3)$ for some $r>0$, $\frac{2}{p}+\frac{3}{q^*}\le 1$, $p\in[2,\infty], q^*\in(3,\infty]$,
(ii) $u\in L^\infty(t_1,t_2;L^3(U_r)^3)$ and $\|u\|_{L^\infty(t_1,t_2;L^3(U_r)^3)}$ is sufficiently small.
Let $\zeta>0$ be such a number that $t_1+\zeta<t_2-\zeta$. Then $u\in L^\infty(t_1+\zeta,t_2-\zeta;W^{2+\delta,2}(U_\rho)^3)$ and both $\frac{\partial u}{\partial t}$ and ∇p belong to $L^\infty(t_1+\zeta,t_2-\zeta;W^{\delta,2}(U_\rho)^3)$ for each $\delta\in[0,\frac{1}{2})$ and $\rho\in(0,r)$.

It follows from Theorem 0.22 that $u\in L^\infty(U_\rho\times(t_1+\zeta,t_2-\zeta))$ and there are no singular points near or on the boundary $\partial\Omega$.

We show that the same result as in Theorem 0.22 can be proved if the conditions (i), (ii) are replaced by conditions on the vorticity field ω. To this purpose we work with a weak suitable solution of (1)–(4) from Theorem 0.4. We will prove the following theorem:

Theorem 0.23. *Let (u,p) be a suitable weak solution of (1) - (4) that satisfies the generalized energy inequality (14) for every $\phi\in C^\infty(\overline{Q}_T), \phi\ge 0$ and every $0<t_1\le t_2<T$. Let*

$$\omega\in L^p(t_1,t_2;L^q(U_R)) \text{ for some } R>0,\ \frac{2}{p}+\frac{3}{q}\le 2,\ p\in[2,\infty], q\in[\frac{3}{2},3]. \tag{62}$$

If $p=\infty$ and $q=\frac{3}{2}$ we still suppose that the norm ω in the space $L^\infty(t_1,t_2;L^{\frac{3}{2}}(U_R))$ is sufficiently small.

Then either the condition (i) or the condition (ii) from Theorem 0.22 is fulfilled for any $0<r<R$ and thus all the conclusions of Theorem 0.22 hold. Especially, there exist no singular points of u in $\overline{U}_r\times(t_1-\zeta,t_2+\zeta)$ for any $r\in(0,R)$ and $\zeta>0$.

Proof: Let $0<r<\eta<R$. We start with the equality (29) with U_η instead of B_2 which holds for every $x\in U_r$. Moreover, η can be chosen in such a way that there are no singular points of u on $(\partial U_\eta\cap\Omega)\times(0,T)$. It follows from the fact that u is a suitable weak solution - for a detailed discussion see [**16**], [**17**], [**18**]. The boundary of U_η consists of two parts, $\partial\Omega$ and $\partial U_\eta\cap\Omega$. Let us investigate firstly the boundary integrals in (29) over $\partial\Omega$. The second integral is equal to zero due to the homogeneous boundary conditions (3). After short computation the first integral can be written as

$$\int_{\partial\Omega}\frac{1}{|x-\xi|}\nabla u(\xi)\cdot n(\xi)\ d_\xi S. \tag{63}$$

Let $\xi\in\partial\Omega$ is an arbitrary point. Then $\frac{\partial u_j}{\partial x_i}(\xi)n_k(\xi)$ is a tensor of the third order and therefore $\frac{\partial u_j}{\partial x_i}(\xi)n_j(\xi)$ is a vector. Let us choose a new coordinate system with the x_1 axis in the direction of the outer normal vector to $\partial\Omega$ in the point ξ. Then the vector $\frac{\partial u_j}{\partial x_i}(\xi)n_j(\xi)$ has the form $(\frac{\partial u_1}{\partial x_1},0,0)$ due to the homogeneous boundary conditions (3) and the fact that in the new coordinate system $n_2(\xi)=n_3(\xi)=0$. Using the continuity equation (2) and the homogeneous boundary conditions (3) we obtain $\frac{\partial u_1}{\partial x_1}(\xi) = -\frac{\partial u_2}{\partial x_2}(\xi)-\frac{\partial u_3}{\partial x_3}(\xi)=0$. Thus, $\frac{\partial u_j}{\partial x_i}(\xi)n_j(\xi)$ is a zero vector in any coordinate system. We can conclude that the integral (63) is equal to zero and the equality (29) can be written as

$$\begin{aligned} u(x) &= \frac{1}{4\pi}\int_{U_\eta} \nabla_\xi \frac{1}{|x-\xi|} \times \omega(\xi)\ d\xi \\ &+\Big\{\frac{1}{4\pi}\int_{\partial U_\eta \cap \Omega} \frac{1}{|x-\xi|}\Big(\omega(\xi)\times n(\xi) + \frac{\partial u}{\partial n}(\xi)\Big)\ d_\xi S \\ &-\frac{1}{4\pi}\int_{\partial U_\eta\cap\Omega} \frac{\partial}{\partial n_\xi}\frac{1}{|x-\xi|} u(\xi)\ d_\xi S\Big\} = u_1(x) + u_2(x). \end{aligned} \tag{64}$$

Since $u \in C^\infty(\overline{\Omega})$ for almost every $t \in (0,T)$ (see e.g. [**6**], Theoreme de Structure), the equality (64) holds for almost every $t \in (t_1, t_2)$.

Now, since $\frac{1}{|x-\xi|} \leq \frac{1}{\eta - r}$ for every $x \in U_r$ and every $\xi \in \partial U_\eta \cap \Omega$ and u is bounded on $(\partial U_\eta \cap \Omega) \times (t_1, t_2)$, we have that

$$u_2 \in L^\infty(U_r \times (t_1, t_2)). \tag{65}$$

For the first integral in (64) we use the result from [**8**], Lemma 7.12, and get that $u_1 \in L^p(t_1, t_2; L^{q^*}(U_r))$, where $\frac{1}{q} - \frac{1}{q^*} = \frac{1}{3}$ and $\frac{2}{p} + \frac{3}{q^*} = 1$. Therefore, if $\frac{3}{2} < q \leq 3$ and $3 < q^* \leq \infty$, then u_1 and also u (see (65)) satisfy the condition (i) from Theorem 0.22. If $q = \frac{3}{2}$ and $q^* = 3$, then u_1 satisfies the condition (ii) from Theorem 0.22. The proof of Theorem 0.23 can now be concluded by the use of (65) and Theorem 0.17. •

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V květnu 1979 jsem složením maturitní zkoušky zakončil studium na gymnáziu v Benešově a téhož roku jsem začal studovat na Matematicko-fyzikální fakultě University Karlovy v Praze. Studium jsem ukončil v roce 1984 složením státní závěrečné zkoušky ve studijním oboru matematická analýza a obhajobou diplomové práce na téma "Geometrické principy funkcionální analýzy, jejich zobecnění a použití" pod vedením Doc. RNDr. J. Daneše, CSc. V roce 1985 jsem na Matematicko-fyzikální fakultě University Karlovy získal titul RNDr. Na téže fakultě jsem pak v roce 1993 obhájil kandidátskou disertační práci na téma "Pomalé proudění vazké nestlačitelné kapaliny s volnou hranicí" pod vedením Doc. RNDr. J.Staré, CSc. a získal jsem vědeckou hodnost kandidáta fyzikálně-matematických věd v oboru matematická analýza.

V září 1984 jsem nastoupil na studijní pobyt do Ústavu pro hydrodynamiku ČSAV a od března 1986 jsem zde byl zaměstnán jako odborný pracovník v oddělení teoretické hydrodynamiky a automatizace experimentálních prací. Zabýval jsem se metodou konečných prvků a jejími aplikacemi při teoretických výpočtech proudění ve velkých vodních nádržích. Od roku 1993 jsem začal pracovat jako vědecký pracovník v oddělení biomechaniky Ústavu pro hydrodynamiku. Používal jsem metodu konečných prvků na výpočty proudění kapalin ve vrapovaných trubicích s aplikací na proudění krve v umělých cevních náhradách. V oddělení biomechaniky jsem se podílel na řešení několika grantů AV ČR a GAČR.

Od října 1999 pracuji na plný pracovní úvazek jako odborný asistent na Katedře matematiky Fakulty stavební ČVUT, kde jsem externě cvičil matematiku již v letech 1993 a 1998. Kromě své pedagogické činnosti jsem spoluřešitelem Výzkumného záměru MSM 210000010 a zabývám se matematickou teorií Navierových-Stokesových rovnic. V Ústavu pro hydrodynamiku AV ČR pracuji od roku 1999 nadále jako vědecký pracovník na částečný pracovní úvazek. V letech 1998 - 2002 jsem se zde podílel na řešení grantu AV ČR "Analýza vybraných smykových toků", od roku 2003 se podílím na řešení grantu AV ČR "Vírový charakter smykových toků". V letech 1999 - 2003 jsem byl řešitelem ústavních úkolů na téma regularity Navierových-Stokesových rovnic.

V devadesátých letech jsem absolvoval několik krátkodobých studijních pobytů ve Velké Británii a to tři studijní pobyty na "University of Manchester Institute of Science and Technology (UMIST)" v letech 1993, 1995 a 1996 u Dr. R.W.Thatchera a studijní pobyt na "School of Mathematics, University of Leeds" u Prof. T.J.Pedleyho v roce 1994.

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